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SOME CHARACTERIZATIONS OF THE PREIMAGE OF A_∞ FOR THE HARDY-LITTLEWOOD MAXIMAL OPERATOR AND CONSEQUENCES

Abstract

The purpose of this paper is to give some characterizations of the weight functions w such that $Mw \in A_\infty(\mathbb{R}^n)$. We show that, for these Mw weights, being in A_∞ ensures being in A_1. We give a criterion in terms of the local maximal functions m_λ and we present a pair of applications, one of them similar to the Coifman-Rochberg characterization of A_1 but using functions of the form $\left(f^{\#}\right)^\delta$ and $(m_\lambda u)^\delta$ instead of $(Mf)^\delta$.

1 Introduction

In this work we look at some characterizations of the weights u in $\mathbb{R}^n$ such that $Mu \in A_\infty$. This question is mentioned as open in [5] and that paper refers the reader to [4] for partial results for monotonic functions in $\mathbb{R}$, and to our knowledge no previous work brings explicitly a complete result. We will show that if for a weight u we have that $Mu \in A_\infty$, actually we have that $Mu \in A_1$. From a result due to Neugebauer it is known that these weights can be characterized for a pointwise condition for the maximal operator: $(M(u^r)(x))^{\frac{1}{r}} \leq CMu(x)$ for some $C > 0, r > 1$ and $\forall x \in \mathbb{R}^n$ a.e., so it is immediately satisfied for a weight belonging to any reverse Hölder class -this means that $\left(u_Q^r\right)^{\frac{1}{r}} \leq C(u_Q)$

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for some $C > 0$, $r > 1$ and for every cube Q with sides parallel to the coordinate axes, where u_Q means the mean of u in Q. Let's remark that in $\mathbb{R}^n$ the pointwise condition, $(M(u^r)(x))^{\frac{1}{r}} \leq CMu(x)$ for a.e. x with a fixed $C > 0$, is strictly weaker than asking for a Gehring condition: $\exists C > 0$ such that $\left(\frac{1}{|Q|}\int_Q u^r\right)^{\frac{1}{r}} \leq C\frac{1}{|Q|}\int_Q u$ for every cube Q. For instance any weight u belonging to $weak - A_\infty$ but not in A_∞ satisfies Neugebauer's condition but for such u there is no $C > 0$ such that $\left(\frac{1}{|Q|}\int_Q u^r\right)^{\frac{1}{r}} \leq C\frac{1}{|Q|}\int_Q u$ for every cube Q. This situation contrasts with the case of local maximal functions -that means: restricted to a finite open cube Q_0- where Neugebauer condition for a.e. $x \in Q_0$ is actually equivalent to a Gehring condition with some fixed constant $C > 0$ for all $Q \subset Q_0$. We refer the reader to reference [2] for that case with local M.

We will also present another condition in terms of the size of sub-level sets, by means of the use of some useful pointwise inequalities found by A. Lerner, involving the sharp maximal operator $u^{\#}$, the local maximal function $m_\lambda(u)$ and the Hardy-Littlewood maximal operator Mu. The resulting condition is weaker than some similar conditions that characterize A_∞ weights. An interesting consequence that we can obtain from this result is a characterization of the A_1 weights similar to the construction of Coifman and Rochberg (which is given in terms of $k(x)(Mf(x))^\delta$ -with k and k^{-1} belonging to L^∞), but involving $u^{\#}$ and $m_\lambda(u)$ instead of $Mf(x)$. As another consequence we can improve, for those weights u such that $Mu \in A_\infty$ -and hence $Mu \in A_1$-, some known inequalities for singular integral operators.

The weights belonging to A_∞ can be described by several conditions. In the reference [7] many of these conditions are enumerated; all of them are mutually equivalent for the usual Muckenhoupt weights for the maximal operator associated with the bases of cubes whose sides are parallel to the coordinate axes (or associated with balls) in $\mathbb{R}^n$, but those that can provide different classes of weights for other bases. Here we deal with the usual bases of cubes (with sides parallel to the coordinate axes) and the corresponding Muckenhoupt weights, but one might translate some of the results for several other bases for which the definitions for A_∞ remain equivalent and for which the properties relating the weights and their A_p constants still hold.

Summarizing the main results are:

Proposition 1. *If u is any weight, $Mu \in A_\infty \iff Mu \in A_1$.*

Theorem 2. *Let u be a weight function in $\mathbb{R}^n$, then $Mu \in A_\infty$ if and only if there exists $s > 1$ and $C_0 > 0$ such that $(Mu^s)^{\frac{1}{s}}(x) \leq C_0 Mu(x)$.*

Criterion 3. *Let u be a weight function, then $Mu \in A_\infty$ if and only if for any $\lambda \in (0,1)$ it holds that $m_\lambda (Mu) \approx M(Mu)$.*

Theorem 4. *Let u be a weight function. Then:*

$$Mu \in A_\infty \iff \exists \alpha > 0, \beta \in (0,1) : \left| \{ y \in Q_x : Mu(y) \leq \alpha (Mu)_Q \} \right| \leq \beta |Q_x|$$

for almost every $x \in \mathbb{R}^n$ for some cube $Q_x \ni x$, and for every cube Q to which x belongs.

Theorem 5.

(1) If $0 < \delta < 1$, $f \in L^1_{loc}(\mathbb{R}^n)$ and $u \in A_1$ and C_1, C_2 non-negative constants then $C_1 \left(f^\# (x) \right)^\delta + C_2 (m_\lambda u(x))^\delta \in A_1$.

(2) Conversely, if $w \in A_1$ then there are $f \in L^1_{loc}(\mathbb{R}^n)$, $u \in A_1$, non-negative constants C_1 and C_2, and $k(x)$with $k, k^{-1} \in L^\infty$ such that $w(x) = k(x) \left(C_1 f^\# (x)^\delta + C_2 m_\lambda u(x)^\delta \right)$.

2 Preliminaries

Here M is the (non-centered) Hardy-Littlewood maximal operator for the bases of cubes with sides parallel to the co-ordinate axes; so if $f \in L^1_{loc}(\mathbb{R}^n)$ we have:

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q f(z)\, dz.$$

A weight w is a non-negative locally integrable function in $\mathbb{R}^n$. A weight $w \in A_p$ class for $1 < p < \infty$ if and only if

$$[A_p] := \sup_Q \left(\frac{1}{|Q|} \int_Q w \right) \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}} \right)^{p-1} < +\infty.$$

A weight $w \in A_1$ if and only if

$$Mw(x) \leq Cw(x) \;\; a.e. x \in \mathbb{R}^n$$

and $[A_1]$ is the minimal constant C such that this inequality occurs.

We will note $f(Q) = \int_Q f(x)\, dx$ and $f_Q = \frac{f(Q)}{|Q|}$.

We also recall the statement of an useful result due to Coifman, R. and Rochberg, R. in characterizing A_1 weights:

Theorem.

(1) Let $f \in L^1_{loc}(\mathbb{R}^n)$ be such that $Mf(x) < \infty$ a.e. and $0 \leq \delta < 1$, then $w(x) = Mf(x)^{\delta}$ is in A_1. Also the A_1 constant depends only on δ.

(2) Conversely, if $w \in A_1$ then there are $f \in L^1_{loc}(\mathbb{R}^n)$ and $k(x)$ with k and k^{-1} both belonging to L^{∞} such that $w(x) = k(x)\, Mf(x)^{\delta}$.

The proof can be found in [6] (or see [3] for the original work), using a suitable decomposition of f and Kolmogorov's inequality for proving (1). The point (2) is quite elementary.

We collect some known properties that we will use. The first three of them can be easily obtained using the definition of A_p classes and the definition of $[A_p]$ constants, and Hölder's inequality (see [6], for instance):

A) $A_p \subset A_q$ if $p < q$ and $[w]_{A_q} \leq [w]_{A_p}$.

B) $w \in A_p$ if and only if $w^{\frac{1}{1-p}} \in A_{\frac{1}{1-p}}$.

C) If $w_0, w_1 \in A_1$ then $w_0 w_1^{1-p} \in A_p$.

Another property that we will need is the reciprocal of property C). That property (P. Jones' Factorization Theorem) it's very much deeper than the previous (see for instance [17]).

D) If $w \in A_p$ there exists $w_0, w_1 \in A_1$ such that $w = w_0 w_1^{1-p}$.

Finally, one last property that we will need is:

E) If $w \in A_p$ there is $\alpha > 1$ such that $w^{\alpha} \in A_p$.

This latter property is usually proved by means of the use of reverse Hölder inequalities that A_p weights satisfy (see [6], [8] or [10]), but it can be obtained easily from the Coifman-Rochberg construction, something perhaps underlooked: if $w \in A_1$ by (2) is $w(x)^{\alpha} = k(x)^{\alpha}\, Mf(x)^{\delta\alpha}$ and taking $1 < \alpha < \frac{1}{\delta}$ we have from (1) that $Mf(x)^{\delta\alpha} \in A_1$ and then

$$
\begin{aligned}
Mw(x)^{\alpha} &\leq M\left(\|k\|_{\infty}^{\alpha}\, Mf(x)^{\delta\alpha}\right) \\
&\leq [(Mf)^{\delta}]_{A_1} \|k\|_{\infty}^{\alpha} \left(Mf(x)^{\delta\alpha}\right) \\
&\leq [(Mf)^{\delta}]_{A_1} \|k\|_{\infty}^{\alpha} \left\|k^{-1}\right\|_{\infty}^{\alpha} k(x)^{\alpha}\, Mf(x)^{\delta\alpha} \\
&= [(Mf)^{\delta}]_{A_1} \|k\|_{\infty}^{\alpha} \left\|k^{-1}\right\|_{\infty}^{\alpha} w(x)^{\alpha}.
\end{aligned}
$$

So $w(x)^\alpha \in A_p$ with $[w]_{A_p} \leq [(Mf)^\delta]_{A_1} \|k\|_\infty^\alpha \left\|k^{-1}\right\|_\infty^\alpha$. On the other hand, for $p>1$ and $w \in A_p$ by property D) we have $w = w_0 w_1^{1-p}$ with $w_0, w_1 \in A_1$ and for $j=0,1$ we write $w_j(x) = k_j(x) Mf_j(x)^{\delta_j}$ and for $1 < \alpha < \min\left\{\frac{1}{\delta_j}\right\}$ we have that $w_0^\alpha, w_1^\alpha \in A_1$ and using C) we have that $w^\alpha = w_0^\alpha (w_1^\alpha)^{1-p} \in A_p$.

By property A, the A_p classes are nested, so it is well defined the class $A_\infty = \bigcup_{p<\infty} A_p$.

A characterization of a weight w for belonging to A_∞ is the following:

$$w \in A_\infty \iff \exists \alpha, \beta \in (0,1) : |\{y \in Q : w(y) \leq \alpha w_Q\}| \leq \beta |Q| \tag{1}$$

for every cube Q (see for instance [7] for this and other characterizations for general bases).

We will prove that for a weight u there is a necessary and sufficient condition for Mu to belong to A_∞ with a statement weaker but quite similar to (1):

$$Mu \in A_\infty \iff \exists \alpha > 0, \beta \in (0,1) : \left|\{y \in Q_x : Mu(y) \leq \alpha (Mu)_Q\}\right| \leq \beta |Q_x| \tag{2}$$

for $x \in \mathbb{R}^n$ a.e. and for some cube $Q_x \ni x$, and for every cube Q to which x belongs.

From the the definition of A_p weights it easily follows that if $u \in A_p$ then either u is locally integrable or $u = \infty$ a.e.

As we call weights to locally integrable non-negative functions, and we want to describe those weights w such that Mw is an A_∞ **weight** we assume that we are always dealing with weights w such that Mw is locally integrable and then $Mw < \infty$ a.e. although on some occasion we neglect to mention it explicitly.

3 Some Results

The first step is the following proposition which shows that if $Mu \in A_\infty$ indeed $Mu \in A_1$, and then because $A_1 \subset A_\infty$ we have that $Mu \in A_\infty \iff Mu \in A_1$. So, what we have to do is to characterize the weights u such that $Mu \in A_1$.

Of course $A_1 \subsetneqq A_\infty$, so there are weights w such that $w \in A_\infty$ and $w \notin A_1$. The lemma tells us that being in A_∞ is the same as being in A_1 for those weights w such that $w = Mu$ for some weight u.

Proof of Proposition 1. The implication $Mu \in A_1 \implies Mu \in A_\infty$ is trivial because $A_1 \subset A_\infty$. It remains to show that if $Mu \in A_\infty \implies Mu \in A_1$.

If $Mu \in A_\infty = \bigcup_{p<\infty} A_p$, we have that $Mu \in A_p$ for some $p \geq 1$. If $p = 1$ there is nothing to prove. Let $p > 1$. Because the result of Coifman and Rochberg we have that $(Mu)^\delta \in A_1$ for any δ with $0 \leq \delta < 1$ and any u locally integrable but generally does not occur that $Mu \in A_1$. Actually we are in the process of proving that if we additionally have that $Mu \in A_p$, in fact $Mu \in A_1$.

We need the following result (see, for instance, [16], ej 5 d) Chap 3): For a measure space (Ω, μ) with measure $\mu(\Omega) = 1$ and $\left(\int_\Omega |f|^r \, d\mu\right)^{\frac{1}{r}} < \infty$ for some $r > 0$, we have that

$$\lim_{r \to 0^+} \left(\int_\Omega |f|^r \, d\mu\right)^{\frac{1}{r}} = \exp\left(\int_\Omega \log(|f|) \, d\mu\right).$$

Let's observe that using that $\mu(\Omega) = 1$ and the Hölder Inequality we obtain $\left(\int_\Omega |f|^{r_1} \, d\mu\right)^{\frac{1}{r_1}} \geq \left(\int_\Omega |f|^{r_2} \, d\mu\right)^{\frac{1}{r_2}}$ if $r_1 \geq r_2$. So for $r > 0$ we have that

$$\left(\int_\Omega |f|^r \, d\mu\right)^{\frac{1}{r}} \geq \exp\left(\int_\Omega \log(|f|) \, d\mu\right) = \lim_{r \to 0^+} \left(\int_\Omega |f|^r \, d\mu\right)^{\frac{1}{r}}.$$

Now for every $q > p$, using that

$$\sup_Q \frac{Mu(Q)}{|Q|} \left(\frac{1}{|Q|} \int_Q Mu(x)^{-\frac{1}{q-1}} \, dx\right)^{q-1} = [Mu]_{A_q} \leq [Mu]_{A_p}$$

(property A), we obtain that for any cube Q :

$$\frac{Mu(Q)}{|Q|} \left(\frac{1}{|Q|} \int_Q Mu(x)^{-\frac{1}{q-1}} \, dx\right)^{q-1} \leq [Mu]_{A_p} < \infty.$$

If q tends to infinity then $\frac{1}{q-1}$ tends to 0^+, so taking $r = \frac{1}{q-1}$ and applying the result from above for $f = w^{-1}$, $\Omega = Q$, and $d\mu = \frac{dx}{|Q|}$, we have

$$\begin{aligned}
\lim_{q \to +\infty} \left(\frac{1}{|Q|} \int_Q Mu(x)^{-\frac{1}{q-1}} \, dx\right)^{q-1} &= \exp\left(\int_Q \log\left(Mu(x)^{-1}\right) dx\right) \\
&= \exp\left(\int_Q -\log(Mu(x)) \, dx\right) \\
&= \frac{1}{\exp\left(\int_Q \log(Mu(x)) \, dx\right)}.
\end{aligned}$$

Taking limit in $\frac{Mu(Q)}{|Q|}\left(\frac{1}{|Q|}\int_Q Mu\,(x)^{-\frac{1}{q-1}}\,dx\right)^{q-1} \leq [Mu]_{A_p}$ we have that

$$\frac{Mu(Q)}{|Q|}\frac{1}{\exp\left(\int_Q \log\left(Mu\,(x)\right)dx\right)} \leq [Mu]_{A_p}$$

so

$$\frac{Mu(Q)}{|Q|} \leq [Mu]_{A_p}\ \exp\left(\int_Q \log\left(Mu\,(x)\right)dx\right).$$

Additionally, the observation from above applied for $f = Mu$ gives us that for any $r > 0$ it holds that

$$\left(\frac{1}{|Q|}\int_Q (Mu)^r\,dx\right)^{\frac{1}{r}} \geq \exp\left(\int_Q \log\left(Mu\,(x)\right)dx\right).$$

Thus

$$\frac{Mu(Q)}{|Q|} \leq [Mu]_{A_p}\ \exp\left(\int_Q \log\left(Mu\,(x)\right)dx\right) \leq [Mu]_{A_p}\left(\frac{1}{|Q|}\int_Q |Mu|^r\,dx\right)^{\frac{1}{r}}$$

and then

$$\frac{Mu(Q)}{|Q|} \leq [Mu]_{A_p}\left(\frac{1}{|Q|}\int_Q |Mu|^r\,dx\right)^{\frac{1}{r}}.$$

Taking $r = \delta$ with $0 \leq \delta < 1$ and using that for such δ it holds that $(Mu)^r = (Mu)^\delta \in A_1$ and then

$$\frac{1}{|Q|}\int_Q |Mu|^r\,dx \leq [(Mu)^r]_{A_1}\ (Mu\,(x))^r$$

a.e for every $x \in Q$.

So we have a.e for $x \in Q$

$$\begin{aligned}\frac{Mu(Q)}{|Q|} &\leq [Mu]_{A_p}\left(\frac{1}{|Q|}\int_Q |Mu|^r\,dx\right)^{\frac{1}{r}}\\ &\leq [(Mu)^r]_{A_1}\ \left([(Mu)^r]_{A_1}\ (Mu\,(x))^r\right)^{\frac{1}{r}}\\ &= [(Mu)^r]_{A_1}\ \left([(Mu)^r]_{A_1}\right)^{\frac{1}{r}}\ (Mu\,(x)).\end{aligned}$$

Taking $C = [(Mu)^r]_{A_1}\ \left([(Mu)^r]_{A_1}\right)^{\frac{1}{r}}$ independent of Q, for every Q we obtain that

$$\frac{Mu(Q)}{|Q|} \leq C\ Mu\,(x)$$

a.e for $x \in Q$.

Then almost everywhere for $x \in \mathbb{R}^n$ we have that

$$M\left(Mu\right)(x) = \sup_{Q \ni x} \frac{Mu(Q)}{|Q|} \leq C\ Mu\,(x)$$

that is

$$M\left(Mu\right)(x) \leq C\ Mu\,(x)$$

and then we obtain that $Mu \in A_1$. □

The previous proposition, together with a lemma due to Neugebauer (published in [4]), enables us to give a characterization of all the weights u such that $Mu \in A_\infty$. Until a few years ago this was an open problem with interesting consequences for improving some two-weight inequalities for several operators, including maximal, vector-valued an Calderon-Zygmund ones (see [5]).

For completeness we transcribe below the lemma of Neugebauer and its easy proof. In [4] the lemma is considered in $\mathbb{R}$ but it works, mutatis mutandis, for $\mathbb{R}^n$.

Lemma. *(Neugebauer) For a weight u it holds that $Mu \in A_1$ if and only if there exists $s > 1$ and $C_0 > 0$ such that $\left(Mu^s\right)^{\frac{1}{s}}(x) \leq C_0 Mu\,(x)$.*

Proof. If such $s > 1$ exists then $\frac{1}{s} < 1$ and the Coifman-Rochberg characterization of A_1 weights tells us that $\left(Mu^s\right)^{\frac{1}{s}}$ is in A_1, so $M\left(\left(Mu^s\right)^{\frac{1}{s}}\right) \leq C_1\left(Mu^s\right)^{\frac{1}{s}}$. Using the hypothesis and the fact that by Hölder: $Mu \leq \left(Mu^s\right)^{\frac{1}{s}}$, we obtain $M\left(Mu\right) \leq M\left(\left(Mu^s\right)^{\frac{1}{s}}\right) \leq C_1\left(Mu^s\right)^{\frac{1}{s}} \leq C_1 CMu$, and then $M(Mu) \leq CMu$, that is $Mu \in A_1$.

Reciprocally, if $Mu \in A_1$ then Mu satisfies a reverse Hölder inequality (RHI), that means that for some $s > 1$ and $C > 0$ it holds for any cube Q

$$\left(\frac{1}{|Q|}\int_Q Mu^s\right)^{\frac{1}{s}} \leq C\frac{1}{|Q|}\int_Q Mu$$

and taking suprema over the cubes we have:

$$\left(Mu^s\right)^{\frac{1}{s}} \leq CMu.$$

□

As we have already mentioned from Neugebauer's lemma together with the Proposition 1, which we recall establishes that $Mu \in A_\infty$ if and only if $Mu \in A_1$, we obtain Theorem 2, whose statement we remember: Let u a weight function in $\mathbb{R}^n$, $Mu \in A_\infty$ if and only if there exists $s > 1$ and $C_0 > 0$ such that $(Mu^s)^{\frac{1}{s}}(x) \leq C_0 Mu(x)$.

PROOF. It is immediate from Neugebauer's lemma and Proposition 1. □

Remark 6. *Let's observe that we have got a bound for the constant* $[Mu]_{A_1}$*, that is*

$$[Mu]_{A_1} \leq [(Mu)^r]_{A_1} \left([(Mu)^r]_{A_1}\right)^{\frac{1}{r}}.$$

Remark 7. *Because of the previous results, the weights* u *with* Mu *in* A_∞ *are those for which there are some* $C > 0$ *such that*

$$M(Mu)(x) \leq CMu(x) \, a.e.$$

4 Some Further Definitions and Properties

Now we will use some pointwise inequalities for certain maximal operators to weaken the above condition. We need a couple of definitions:

Definition 8. *If* $f \in L^1_{loc}(\mathbb{R}^n)$ *the sharp maximal function of Fefferman-Stein* $f^\#$ *is defined by*

$$f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx.$$

Definition 9. $BMO(\mathbb{R}^n) = \{f \in L^1_{loc}(\mathbb{R}^n) : f^\# \in L^\infty(\mathbb{R}^n)\}$ *is the space of functions with bounded mean oscillation, and* $\|f\|_{BMO} = \left\|f^\#\right\|_\infty$.

Let's notice that $\|\|_{BMO}$ is a seminorm for $BMO(\mathbb{R}^n)$ since $\left\|f^\#\right\|_\infty = 0$ if and only if f is constant (a.e.). It is usual to identify BMO with its quotient with the class of almost everywhere constant functions and then $\|\|_{BMO}$ becomes a norm.

Notation 10. *For a measurable function* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$*, the non-increasing rearrangement of* f *is* f^**. That is, for* $t \geq 0$

$$f^*(t) = \inf\{\alpha > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}| \leq t\}.$$

We use the convention that $\inf \emptyset = \infty$*.*

An equivalent way to define $f^*(t)$ is

$$f^*(t) = \sup_{|E|=t} \inf_{x\in E} |f(x)|$$

where E are measurable sets.

Remark 11. *Non-increasing rearrangements of functions from measure spaces (X,μ) can be defined in the same way replacing $\mathbb{R}^n$ by X and the Lebesgue measure $|\ |$ by μ. Much more details and results can be found in [1].*

Definition 12. *If f is a measurable function and $\lambda \in (0,1)$ the local maximal functions $m_\lambda(f)$ are defined by*

$$m_\lambda f(x) = \sup_{Q\ni x} (f\chi_Q)^* (\lambda |Q|) .$$

Let's point out some basic properties of f^*, $m_\lambda f(x)$, and $f^\#$, immediate from their definitions:

(i) $f^\#(x) \le 2Mf(x)$.

(ii) If $c > 0$ then $(cf)^*(t) = c(f)^*(t)$.

(iii) If $f(x) \ge g(x)$ *a.e.* then $f^*(t) \ge g^*(t)$ for every t.

(iv) Using (iii) if $f(x) \ge g(x)$ *a.e.* then $m_\lambda(f)(x) \ge m_\lambda(g)(x)$ everywhere.

(v) If $c > 0$ using ii) we have $m_\lambda(cf)(x) = cm_\lambda(f)(x)$.

We will also need the somewhat less trivial inequalities:

Lemma 13 (vi)**.** *$m_\lambda(f)(x) \ge |f(x)|$ that holds at every Lebesgue point of f, so a.e. if $f \in L^1_{loc}(\mathbb{R}^n)$.*

Proof. We will need to remember a definition and a known result of Real Analysis. The definition is the following: a sequence $\{E_i\}_{i\in\mathbb{N}}$ of Borel sets of $\mathbb{R}^n$ is said to shrink to x nicely if there is a number $\alpha > 0$ such that there is a sequence of cubes of $\mathbb{R}^n$ centered at x of radii $r_i \to 0$, $\{Q_{(x,r_i)}\}_{i\in\mathbb{N}}$, such that $E_i \subset Q_{(x,r_i)}$ and $|E_i| \ge \alpha \left|Q_{(x,r_i)}\right|$. The result is this: if $x \in \mathbb{R}^n$ is a Lebesgue point of $f \in L^1_{loc}(\mathbb{R}^n)$ and $\{E_i\}_{i\in\mathbb{N}}$ is a sequence of sets that shrinks to x nicely, then

$$f(x) = \lim_{i\to\infty} \frac{1}{|E_i|} \int_{E_i} f(z)\, dz$$

(see [Rudin], Theorem 7.10 – changing cubes for balls and $f \in L^1_{loc}(\mathbb{R}^n)$ instead of $f \in L^1(\mathbb{R}^n)$ the proof still works).

Now for any positive τ with $\tau < 1$, using the definitions of non-increasing rearrangements and m_λ we have that

$$\forall Q \ni x : |\{y \in Q : |f(y)| > \tau m_\lambda f(x)\}| \geq \lambda |Q| .$$

So if we take $r_i = \frac{1}{i} \to 0$ and we name

$$\{E_i\}_{i\in\mathbb{N}} = \{y \in Q_{(x,r_i)} : |f(y)| \leq \tau m_\lambda f(x)\}$$

then

$$E_i = Q_{(x,r_i)} \setminus \{y \in Q_{(x,r_i)} : |f(y)| > \tau m_\lambda f(x)\}$$

and we obtain that

$$|E_i| = \left|Q_{(x,r_i)} \setminus \{y \in Q_{(x,r_i)} : |f(y)| > \tau m_\lambda f(x)\}\right| \geq Q_{(x,r_i)} - \lambda \left|Q_{(x,r_i)}\right| .$$

That is,

$$|E_i| \geq (1-\lambda) \left|Q_{(x,r_i)}\right|$$

and then $\{E_i\}_{i\in\mathbb{N}}$ is a sequence of sets that shrinks to x nicely. But now, with these sets E_i we can apply the mentioned result for any Lebesgue point to obtain:

$$f(x) = \lim_{i\to\infty} \frac{1}{|E_i|} \int_{E_i} f(z)\, dz \leq \lim_{i\to\infty} \frac{1}{|E_i|} \int_{E_i} \tau m_\lambda f(x)\, dz$$

and using $|f(x)|$ instead of $f(x)$:

$$|f(x)| \leq \lim_{i\to\infty} \frac{\tau m_\lambda f(x)}{|E_i|} \int_{E_i} dz = \lim_{i\to\infty} \frac{\tau m_\lambda f(x)}{|E_i|} |E_i| = \tau m_\lambda f(x) .$$

Then

$$|f(x)| \leq \tau m_\lambda f(x)$$

$\forall \tau < 1$, and taking limit for $\tau \to 1^-$ we obtain:

$$|f(x)| \leq m_\lambda f(x)$$

for every Lebesgue point of f and then almost everywhere. □

We now list a few more properties:

(vii) For any $\lambda \in (0,1)$ there is a constant $c_{\lambda,n}$ (depending only of λ and n) such that for all $u \in L^1_{loc}$ and $x \in \mathbb{R}^n$ we have (see [12] or [13]):

$$m_\lambda (Mu)(x) \leq c_{\lambda,n} u^{\#}(x) + Mu(x) .$$

(viii) Observe that using (vii) and applying (vi) to $f = Mu$ we obtain

$$m_\lambda (Mu)(x) \le cMu(x)$$

a.e. for some $c > 0$.

(ix) $m_\lambda (Mu)$ and Mu are pointwise equivalent a.e.(we will write $m_\lambda (Mu) \approx Mu$ for that situation) that is there are positive constants A and B such that $m_\lambda(Mu)(x) \le AMu(x)$ and $Mu(x) \le Bm_\lambda(Mu(x))$ a.e., we obtain this taking $A = c$ in (viii), and $B = 1$ in (vi).

(x) It's immediate from the definition of M that $Mf(x) \ge f(x)$ a.e.

5 Some More Results

From Theorem 2 we have that $Mu \in A_\infty$ if and only if $Mu \approx M(Mu)$. A weaker statement is actually enough to guarantee that $Mu \in A_\infty$. This is Criterion 3 and it follows from our Proposition 1 and inequality (vii):

Proof. By Proposition 1 we have $Mu \in A_\infty \iff Mu \in A_1$, so $Mu \in A_\infty$ if and only if there is some $C > 0 : M(Mu)(x) \le CMu(x)$ a.e. and using that $M(f)(x) \ge f(x)$ a.e. for $f \in L^1_{loc}$ we have that $M(Mu)(x) \ge Mu(x)$ a.e., and then (ix) gives us that $Mu \in A_\infty \iff Mu \in A_1 \iff Mu \approx M(Mu) \iff m_\lambda(Mu) \approx M(Mu)$. □

Remark 14. *We can observe that it is enough that $m_\lambda(Mu) \approx M(Mu)$ for some $\lambda \in (0,1)$ to obtain that $Mu \in A_\infty$ and then $m_\lambda(Mu) \approx M(Mu)$ for every $\lambda \in (0,1)$.*

Because of (viii) for any u we always can ensure for a suitable $c > 0$ that $m_\lambda(Mu)(x) \le cMu(x) \le cM(Mu)(x)$, that is $m_\lambda(Mu)(x) \le cM(Mu)(x)$ a.e. Thus, by the criterion above, a condition necessary and sufficient, on u, for Mu to belong to A_∞ is the existence of a constant $C > 0$ such that $M(Mu)(x) \le Cm_\lambda(Mu)(x)$ a.e.

As we mentioned in the introduction, now we want to prove that (2) is a necessary and sufficient condition on a weight u for Mu to be in A_∞.

A condition like (2) but applied for an arbitrary weight w instead of Mu is weaker than (1), that is, if $w \in A_\infty$ then w satisfies the following:

Condition 15 (LocalAINF)**.** *$\exists \alpha_1 > 0, \beta_1 \in (0,1)$ such that for almost every $x \in \mathbb{R}^n$ exists a cube $Q_x \ni x$ that $\forall Q \ni x$ verifies that*

$$|\{y \in Q_x : w(y) \le \alpha_1 w_Q\}| \le \beta_1 |Q_x|.$$

To see this implication let's remember that $w \in A_\infty$ if and only if w satisfies:

Condition 16 (CAINF)**.** $\exists \alpha, \beta \in (0,1) : \forall Q$ *cube we have*

$$|\{y \in Q : w(y) \leq \alpha w_Q\}| \leq \beta |Q|.$$

Now, if $w \in A_\infty$ we fix some $k \in (0,1)$, for instance $k = \frac{1}{2}$, and for any x we take a cube $Q_x \ni x$ such that $w_{Q_x} = \frac{w(Q_x)}{|Q_x|} \geq kMw(x)$. So let $\alpha_1 = \alpha k$ and for any $Q \ni x$ we have that

$$\{y \in Q_x : w(y) \leq \alpha_1 w_Q\} \subset \{y \in Q_x : w(y) \leq \alpha_1 Mw(x)\}$$

$$\subset \{y \in Q_x : w(y) \leq \frac{\alpha_1}{k} \frac{w(Q_x)}{|Q_x|}\},$$

then applying the previous condition to Q_x we have

$$|\{y \in Q_x : w(y) \leq \alpha_1 w_Q\}| \leq \left|\{y \in Q_x : w(y) \leq \frac{\alpha_1}{k} w_{Q_x}\}\right|$$

$$= |\{y \in Q_x : w(y) \leq \alpha w_{Q_x}\}| \leq \beta |Q_x|$$

so the condition (LocalAINF) is fulfilled with $\alpha_1 = \alpha k$, $\beta_1 = \beta$ and the Q_x selected for which $\frac{w(Q_x)}{|x|} \geq kMw(x)$.

Then we have that it also holds for the following case:

Although the condition (LocalAINF) is weaker than A_∞ for a general weight when it is applied to a weight that is the maximal function of another weight, that is if $w = Mu$ then the condition (LocalAINF) implies A_∞, so they are equivalent conditions for Mu weights. That is the statement of Theorem 4 which ensures that

$$Mu \in A_\infty \iff \exists \alpha > 0, \beta \in (0,1) : \left|\{y \in Q_x : Mu(y) \leq \alpha (Mu)_Q\}\right| \leq \beta |Q_x|$$

for almost every $x \in \mathbb{R}^n$ for some cube $Q_x \ni x$, and for every cube Q to which x belongs.

Proof. Because of the previous remark, $Mu \in A_\infty$ if and only if there exists a positive constant B and $\lambda \in (0,1)$:

$$M(Mu)(x) \leq Bm_\lambda(Mu(x))a.e. \tag{3}$$

So to guarantee $Mu \in A_\infty$ is equivalent to have:

$$\alpha M(Mu)(x) \leq m_\lambda(Mu(x)) \tag{4}$$

for some $\alpha > 0$ and almost every $x \in \mathbb{R}^n$. Now using the definition of m_λ we have that (4) is equivalent to say that for almost every $x \in \mathbb{R}^n$

$$\exists Q_x \ni x : \left(Mu\chi_{Q_x}\right)^* \left(\lambda \left|Q_x\right|\right) \geq \alpha \left(Mu\right)_Q$$

for every cube $Q \ni x$. Now by the definition of non-increasing rearrangements this means that for a.e. $x \in \mathbb{R}^n$

$$\exists Q_x \ni x : \left|\{y \in Q_x : Mu\left(y\right) > \alpha \left(Mu\right)_Q\}\right| > \lambda \left|Q_x\right|$$

for every cube $Q \ni x$, or, taking complements respect Q_x and naming $\beta = (1-\lambda) \in (0,1)$, we have (3). Therefore $Mu \in A_\infty$ is equivalent to the existence of $\alpha > 0, \beta \in (0,1)$ such that for almost every $x \in \mathbb{R}^n$ there is some $Q_x \ni x$:

$$\exists Q_x \ni x : \left|\{y \in Q_x : Mu\left(y\right) \leq \alpha \left(Mu\right)_Q\}\right| \leq \beta \left|Q_x\right|$$

for every cube $Q \ni x$. □

Example 17. *It's easy to see that a class of weights functions u such that $Mu \in A_\infty$ is the class A_∞ itself, that is $M\left(A_\infty\right) \subset A_\infty$, and by our first proposition in fact $M\left(A_\infty\right) \subset A_1$. Indeed we can provide an elementary proof of this using the previous theorem and the characterization (1) of A_∞ weights: We fix some $k \in (0,1)$, and for any x we take a cube Q_x such that $\frac{Mu(Q_x)}{|Q_x|} \geq kM\left(Mu\right)(x)$; because (1) and the fact that $u \in A_\infty$ we have α_1, β_1 such that for any cube $\widetilde{Q}$ it holds: $\left|\{y \in \widetilde{Q} : u\left(y\right) \leq \alpha_1 u_{\widetilde{Q}}\}\right| \leq \beta_1 \left|\widetilde{Q}\right|$. Then for $\widetilde{Q} = Q_x$, $\alpha = \frac{\alpha_1}{k}$, $\beta = \beta_1$ and for any $Q \ni x$, and using the trivial inclusions due to the inequalities $\frac{Mu(Q_x)}{|Q_x|} \geq kM\left(Mu\right)(x)$; $MMu\left(z\right) \geq Mu\left(z\right)$ a.e. and $Mu\left(z\right) \geq u\left(z\right)$ a.e. we get:*

$$\begin{aligned} \left|\{y \in Q_x : Mu\left(y\right) \leq \alpha \frac{Mu\left(Q\right)}{|Q|}\}\right| &\leq \left|\{y \in Q_x : Mu\left(y\right) \alpha \frac{MMu\left(Q\right)}{|Q|}\}\right| \\ &\leq \left|\{y \in Q_x : Mu\left(y\right) \leq \alpha M\left(Mu\right)(x)\}\right| \\ &\leq \left|\{y \in Q_x : u\left(y\right) \leq \alpha M\left(Mu\right)(x)\}\right| \\ &\leq \left|\{y \in Q_x : u\left(y\right) \leq \frac{\alpha}{k} \frac{Mu\left(Q_x\right)}{|Q_x|}\}\right| \\ &\leq \beta \left|Q_x\right| \end{aligned}$$

that is we have

$$\left|\{y \in Q_x : Mu\left(y\right) \leq \frac{\alpha}{k} \frac{Mu\left(Q\right)}{|Q|}\}\right| \leq \beta \left|Q_x\right|.$$

Example 18. *Actually for those functions there is a shorter way to prove that $Mu \in A_1$: Because of Hölder's inequality we have that for all $r > 1$:*

$$\frac{1}{|Q|}\int_Q u(x) \leq \left(\frac{1}{|Q|}\int_Q u^r(x)\right)^{\frac{1}{r}}$$

and taking suprema

$$Mu(x) \leq (M(u^r)(x))^{\frac{1}{r}}.$$

Now for the Coifman-Rochberg characterization of A_1 weights for any locally integrable function g and $\delta \in [0,1)$ we have that $Mg(x)^\delta \in A_1$ and then $(M(u^r)(x))^{\frac{1}{r}} \in A_1$. Therefore, for some constant $C > 1$:

$$MMu(x) \leq M\left((M(u^r)(x))^{\frac{1}{r}}\right) \leq C(M(u^r)(x))^{\frac{1}{r}}$$

a.e. But if $u \in A_\infty$ then $u \in A_p$ for some $p \geq 1$, and then it satisfies a reverse Hölder inequality (see [6]) for some $r > 1$, that is

$$\left(\frac{1}{|Q|}\int_Q u^r(x)\right)^{\frac{1}{r}} \leq C\frac{1}{|Q|}\int_Q u(x)$$

for certain $C > 0$, thus

$$(M(u^r)(x))^{\frac{1}{r}} \leq CMu(x)$$

and then

$$MMu(x) \leq CMu(x)$$

a.e. That is $Mu \in A_1$.

Remark 19. *We remark that this latter way to prove that $M(A_\infty) \subset A_1$ requires two strong results: the characterization of A_1 and the reverse Hölder inequality for A_p weights, while proposition 1 is elementary.*

Example 20. *A larger class of weights that M sends to A_1 are the weak$-A_\infty$ weights.*

We recall that $u \in A_\infty$ if and only if there exists positive constants C and δ such that for any cube Q and any measurable $E \subset Q$:

$$u(E) \leq C\left(\frac{|E|}{|Q|}\right)^\delta u(Q).$$

Let's give the definition of $weak - A_\infty$ weights: $u \in weak - A_\infty$ if and only if there exists positive constants C and δ such that for any cube Q and any measurable $E \subset Q$:

$$u(E) \leq C \left(\frac{|E|}{|Q|} \right)^{\delta} u(2Q). \tag{5}$$

Remark 21. *Let's note that it's easy to prove that we can replace the factor 2 with any constant $k > 1$, obtaining an equivalent definition of $weak - A_\infty$.*

It's clear that if $u \in A_\infty$ then $u \in weak - A_\infty$ because any $u \in A_\infty$ is a doubling weight (see [6]), that is $u(2Q) \leq Cu(Q)$ for some $C > 0$ and for every cube Q.

It's a known result that an equivalent condition for u to be in A_∞ is to belong to a RHI class, that means that for some $r > 1$ and $C > 0$ it holds for any cube Q

$$\left(\frac{1}{|Q|} \int_Q u^r \right)^{\frac{1}{r}} \leq C \frac{1}{|Q|} \int_Q u.$$

Remark 22. *Let's remark that those weights that belongs to $weak - A_\infty$ but that don't belong to A_∞ are always non-doubling weights.*

A corollary that we can obtain immediately taking suprema on the RHI condition for A_∞ weights is that for any $x \in \mathbb{R}^n$

$$\left(M(u^r)(x) \right)^{\frac{1}{r}} \leq CMu(x).$$

It can be easily obtained for $weak - A_\infty$ weights a condition analogous to RHI, we include the statement and the proof for completeness:

Lemma 23. *If $u \in weak - A_\infty$, then there are some $r > 1$ and $C > 0$ such that for any cube Q*

$$\left(\frac{1}{|Q|} \int_Q u^r \right)^{\frac{1}{r}} \leq C \frac{1}{|2Q|} \int_{2Q} u.$$

PROOF. Let Q be any cube and $E_t = \{x \in Q : u(x) > t\}$. Now, applying the definition of E_t and (5) we have $t|E_t| \leq u(E_t) \leq C\frac{|E_t|^\delta}{|Q|^\delta} u(2Q)$. Hence, using $|2Q| = 2^n |Q|$ and incorporating the factor 2^n to the constant C:

$$t|E_t|^{1-\delta} \leq C|Q|^{1-\delta} \frac{u(2Q)}{|2Q|}$$

so

$$|E_t| \leq C t^{\frac{-1}{1-\delta}} |Q| \left(\frac{u(2Q)}{|2Q|} \right)^{\frac{1}{1-\delta}}.$$

Now we use this inequality in the layer-cake formula. Let's be $k \in (0, \infty)$ that we will choose later:

$$\begin{aligned} \int_Q u^r &= \int_0^\infty r t^{r-1} |E_t| \, dt \\ &= \int_0^\infty r t^{r-1} |E_t| \, dt \\ &= \int_0^k r t^{r-1} |E_t| \, dt + \int_k^\infty r t^{r-1} |E_t| \, dt, \end{aligned}$$

then

$$\int_Q u^r \leq \int_0^k r t^{r-1} |Q| \, dt + C \int_k^\infty r t^{r-1} t^{\frac{-1}{1-\delta}} |Q| \left(\frac{u(2Q)}{|2Q|} \right)^{\frac{1}{1-\delta}} dt,$$

that is:

$$\int_Q u^r \leq |Q| \, t^r \big|_0^k + C |Q| \left(\frac{u(2Q)}{|2Q|} \right)^{\frac{1}{1-\delta}} \frac{r}{r - \frac{1}{1-\delta}} t^{r-\frac{1}{1-\delta}} \Big|_k^\infty.$$

Then, for $r : 1 < r < \frac{1}{1-\delta}$ we get:

$$\frac{1}{|Q|} \int_Q u^r \leq k^r + C \frac{r}{\frac{1}{1-\delta} - r} \left(\frac{u(2Q)}{|2Q|} \right)^{\frac{1}{1-\delta}} k^{r-\frac{1}{1-\delta}}.$$

Now choosing $k = \frac{u(2Q)}{|2Q|}$ it results:

$$\frac{1}{|Q|} \int_Q u^r \leq \left(\frac{u(2Q)}{|2Q|} \right)^r + C \frac{r}{\frac{1}{1-\delta} - r} \left(\frac{u(2Q)}{|2Q|} \right)^{\frac{1}{1-\delta}} \left(\frac{u(2Q)}{|2Q|} \right)^{r-\frac{1}{1-\delta}}$$

hence

$$\frac{1}{|Q|} \int_Q u^r \leq \left(C \frac{r}{\frac{1}{1-\delta} - r} \right) \left(\frac{u(2Q)}{|2Q|} \right)^r,$$

and renaming the constant we have:

$$\left(\frac{1}{|Q|} \int_Q u^r \right)^{\frac{1}{r}} \leq C \frac{u(2Q)}{|2Q|}.$$

□

Corollary 24. *From the previous lemma it's obvious that the pointwise inequality*

$$\left(M\left(u^r\right)(x)\right)^{\frac{1}{r}} \leq CMu\,(x) \tag{6}$$

still remains true for weak $-A_\infty$ *weights and using Neugebauer's Lemma the weights* $u \in weak - A_\infty$ *satisfy that* $Mu \in A_1$.

Actually the condition $\left(\frac{1}{|Q|}\int_Q u^r\right)^{\frac{1}{r}} \leq C\frac{1}{|2Q|}\int_{2Q} u$ characterizes the $weak-A_\infty$ weights; it can be proved that the converse of the previous lemma is also true. Nevertheless, we will not need that result here. As we mentioned in a previous remark we can replace the constant 2 for any $k > 1$, so $u \in weak-A_\infty$ if and only if there exists some positive constant C such that for any $k > 1$ and every cube Q

$$\left(\frac{1}{|Q|}\int_Q u^r\right)^{\frac{1}{r}} \leq C\frac{1}{|kQ|}\int_{kQ} u. \tag{7}$$

We have already seen that $A_\infty \subset weak - A_\infty \subset M^{-1}(A_\infty)$ where we denote $M^{-1}(A_\infty)$ the class of weights u such that $Mu \in A_\infty$.

It's interesting to observe that this question has a close relationship with another one involving the weighted Fefferman-Stein inequality in $L^p(w)$:

$$\|f\|_{L^p(w)} \leq c\left\|f^{\#}\right\|_{L^p(w)} \qquad (1 < p < \infty) \tag{8}$$

for some $c > 0$, and for every $f \in L^p$ such that $f \in S_0(\mathbb{R}^n)$, where $S_0(\mathbb{R}^n)$ is the space of measurable functions f on $\mathbb{R}^n$ such that for any $t > 0$

$$\mu_f(t) = |\{x \in \mathbb{R}^n : |f(x)| > t\}| < \infty.$$

The inequality 8 is equivalent to many interesting others, for instance, with the same hypothesis of 8:

$$\|Mf\|_{L^p(w)} \leq c\left\|f^{\#}\right\|_{L^p(w)} \qquad (1 < p < \infty)$$

or for some $c > 0$, $r > 1$ and for any $f \in L^1_{loc}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \mathcal{M}_{p,r}(f,w)\,|f|\,dx \leq c\int_{\mathbb{R}^n} (Mf)^p\,w dx \qquad (1 < p < \infty) \tag{9}$$

where $\mathcal{M}_{p,r}(f,w) = \sup\limits_{Q \ni x}\left(\frac{1}{|Q|}\int_Q |f|\right)^{p-1}\left(\frac{1}{|Q|}\int_Q w^r\right)^{\frac{1}{r}}$. The equivalence of those inequalities is proven in [14].

Remark 25. *Related to the (to our knowledge) open question about for which weights the former inequalities hold are the following inclusions of nested classes:* $A_\infty \subset weak - A_\infty \subset C_{p+\varepsilon} \subset C_p$ *where* $\varepsilon > 0$ *and* C_p *condition means that there exists* $c, \delta > 0$ *such that for any cube* Q *and any measurable* $E \subset Q$

$$u\left(E\right) \leq c\left(\frac{|E|}{|Q|}\right)^{\delta} \int_{\mathbb{R}^n} \left(M\chi_Q\right)^p u.$$

Remember that for $u \in A_1$*, for any cube* Q *and any measurable* $E \subset Q$

$$u\left(E\right) \leq c\left(\frac{|E|}{|Q|}\right)^{\delta} u\left(Q\right) = c\left(\frac{|E|}{|Q|}\right)^{\delta} \int_{\mathbb{R}^n} \left(\chi_Q\right)^p u$$

and for $weak - A_\infty$ *weights:* $u \in weak - A_\infty$ *if and only if there exist positive constants* C *and* δ *such that for any cube* Q *and any measurable* $E \subset Q$:

$$u\left(E\right) \leq C\left(\frac{|E|}{|Q|}\right)^{\delta} \int_{\mathbb{R}^n} \left(\chi_{2Q}\right)^p u$$

and the mentioned inclusion are obvious. It can be found in [14] (see also [18]) that C_p *is necessary and* $C_{p+\varepsilon}$ *is sufficient for 9 or 8. Also, [14] introduces a new sufficient condition* $\widetilde{C_p}$ *instead of* $C_{p+\varepsilon}$ *but it is not known if* $\widetilde{C_p}$ *or* $C_{p+\varepsilon}$ *are necessary conditions.*

The inclusion relations from $A_\infty \subset weak - A_\infty \subset M^{-1}\left(A_\infty\right)$ *and* $A_\infty \subset weak - A_\infty \subset C_{p+\varepsilon} \subset C_p$ *and the former inequalities seems to be closely linked. For instance,* $u \in C_p$ *is necessary for 9, and 9 implies that for any* Q *we have that* $\left(\frac{1}{|Q|}\int_Q u^r\right)^{\frac{1}{r}} \leq c\frac{1}{|Q|}\int_{\mathbb{R}^n} \left(M\chi_Q\right)^p u$*, which is a bit weaker than* $\left(\frac{1}{|Q|}\int_Q u^r\right)^{\frac{1}{r}} \leq C\frac{1}{|Q|}\int_{\mathbb{R}^n} \left(\chi_{2Q}\right)^p u$ *which is equivalent to* $weak - A_\infty$.

Additionally, in [14] it is proven that C_p *is necessary for*

$$\int_{\mathbb{R}^n} \mathcal{M}_{p,r}\left(f, w\right)|f|\, dx \leq c\int_{\mathbb{R}^n} \left(Mf\right)^p w dx,$$

that is 9 implies C_p.

On the other hand, using the lemma of Neugebauer, which tells that

$$\left(Mu^r\right)^{\frac{1}{r}}\left(x\right) \leq CMu\left(x\right) \text{ for } u \in M^{-1}\left(A_\infty\right) \text{ for some } C > 0, r > 1,$$

and the definition of $\mathcal{M}_{p,r}(f,u)$ we obtain that if $u \in M^{-1}(A_\infty)$ then

$$\begin{aligned}\mathcal{M}_{p,r}(f,w)(u) &= \sup_{Q\ni x}\left(\frac{1}{|Q|}\int_Q |f|\right)^{p-1}\left(\frac{1}{|Q|}\int_Q u^r\right)^{\frac{1}{r}} \\ &\le \sup_{Q\ni x}\left(\frac{1}{|Q|}\int_Q |f|\right)^{p-1} M_r u(x) \\ &\le \sup_{Q\ni x}\left(\frac{1}{|Q|}\int_Q |f|\right)^{p-1} CMu(x) \\ &\le (Mf)^{p-1}(x)\, CMu(x),\end{aligned}$$

and then integrating we have:

$$\int_{\mathbb{R}^n} \mathcal{M}_{p,r}(f,w)\,|f|\,dx \le c\int_{\mathbb{R}^n} (Mf)^p\, Mwdx \tag{10}$$

(compare with 9). So we have that $M^{-1}(A_\infty)$ implies 10 and 9 implies C_p.

6 A Couple of Applications

Using the criterion that $Mu \in A_\infty$ if and only if for any $\lambda \in (0,1)$ it holds that $m_\lambda(Mu) \approx M(Mu)$, we can derive a characterization of the A_1 weights similar to the construction of Coifman and Rochberg.

First of all we introduce the definition of the local sharp maximal operator, which for $0 < \lambda < 1$ we define:

$$M_\lambda^{\#} f(x) = \sup_{Q\ni x}\inf_c\left((f-c)\chi_Q\right)^*(\lambda|Q|).$$

The sharp maximal function has a similar role to the Hardy-Littlewood maximal operator for the local sharp maximal functions because there are positive constants c_1 and c_2 such that for $f \in L^1_{loc}$:

$$c_1 MM_\lambda^{\#} f(x) \le f^{\#}(x) \le c_2 MM_\lambda^{\#} f(x),$$

(see [11]). Using the former inequalities we easily get that for the sharp function an statement similar to the first one of the Coifman-Rochberg theorem:

Lemma 26. *Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $0 \le \delta < 1$, then $w(x) = f^{\#}(x)^\delta$ is in A_1.*

Proof. For

$$c_1^\delta\left(MM_\lambda^{\#} f(x)\right)^\delta \le f^{\#}(x)^\delta \le c_2^\delta\left(MM_\lambda^{\#} f(x)\right)^\delta$$

and $\left(MM_\lambda^{\#}f(x)\right)^\delta \in A_1$ because of the mentioned result of Coifman and Rochberg. Now

$$\begin{aligned} Mf^{\#}(x)^\delta &\leq M\left(c_2^\delta\left(MM_\lambda^{\#}f(x)\right)^\delta\right) \\ &\leq c_2^\delta[MM_\lambda^{\#}f(x)]_{A_1}\left(MM_\lambda^{\#}f(x)\right)^\delta \\ &\leq Cf^{\#}(x)^\delta \end{aligned}$$

with constant $C = \frac{c_2^\delta}{c_1^\delta}[MM_\lambda^{\#}f(x)]_{A_1}$, so $f^{\#}(x)^\delta \in A_1$. □

We don't know if any $w \in A_1$ always could be written as $k(x)f^{\#}(x)^\delta$ for suitable $f \in L^1_{loc}$; $0 < \delta < 1$ and $k, k^{-1} \in L^\infty$, but we can obtain a result similar to the second part of Coifman-Rochberg Theorem if we added a multiple of the local maximal function m_λ:

Proposition 27. *If $w \in A_1$ then there are $k(x)$ such that $k, k^{-1} \in L^\infty$ and a constants $C_1, C_2 > 0$ such that*

$$w(x) = k(x)\left(C_1\left((w^\alpha(x))^{\#}\right)^\delta + C_2\left(m_\lambda w^\alpha(x)\right)^\delta\right).$$

Proof. If $w \in A_1$ we can use the property E) to take $\alpha > 1$ such that $w^\alpha \in A_1$. Thus $M(w^\alpha) \in A_1$. Now for w^α, using the above criterion that establishes that $Mu \in A_1$ if and only if $m_\lambda(Mu) \approx M(Mu)$ and then in such situation: $m_\lambda(M(w^\alpha)) \approx M(M(w^\alpha)) \approx M(w^\alpha) \approx w^\alpha$. Also, we have that $Mw \approx w$ because $w \in A_1$ and also using the pointwise inequalities mentioned in (xi) and (vii): $m_\lambda(Mu)(x) \leq c_{\lambda,n}u^{\#}(x) + Mu(x)$ and $m_\lambda(Mu)(x) \leq c_{\lambda,n}u^{\#}(x) + Mu(x)$, for $u = w^\alpha$ we have:

$$w(x)^\alpha \leq M(w^\alpha)(x) \leq c_{\lambda,n}(w^\alpha)^{\#}(x) + m_\lambda(w^\alpha)(x).$$

Then with $\delta = \frac{1}{\alpha}$ it is $0 < \delta < 1$ and $\alpha\delta = 1$. Also we will use property (i): $u^{\#} \leq 2Mu$ pointwise, properties (vi) ($|f(x)| \leq m_\lambda f(x)$) and (x) ($f(x) \leq Mf(x)$) and that if $f(x) \leq g(x)$ a.e. for positive functions then $Mf(x) \leq Mg(x)$ and $m_\lambda(f)(x) \leq m_\lambda(g)(x)$ a.e.

Furthermore, we use the sublinearity of M and the facts that w^α and w are in A_1, and because of the criterion we can use that for $w \in A_1$ then $Mw \in A_1$ too. It occurs that $m_\lambda(Mw) \approx M(Mw) \approx Mw \approx w$. We will number or rename the constants that appear. Also we will use that $M\left((Mw^\alpha)^\delta\right) \leq$

$C\left(Mw^{\alpha}\right)^{\delta}$ (because $\left(Mf\right)^{\delta}\in A_1$ by Coifman-Rochberg). So we get:

$$\begin{aligned}
w\left(x\right) &\leq \left(c_1\left(w^{\alpha}\right)^{\#}\left(x\right)+m_{\lambda}\left(w^{\alpha}\right)\left(x\right)\right)^{\delta}\\
&\leq c_2\left(\left(w^{\alpha}\right)^{\#}\left(x\right)\right)^{\delta}+\left(m_{\lambda}\left(w^{\alpha}\right)\left(x\right)\right)^{\delta}\\
&\leq M\left(c_2\left(\left(w^{\alpha}\right)^{\#}\left(x\right)\right)^{\delta}+\left(m_{\lambda}\left(w^{\alpha}\right)\left(x\right)\right)^{\delta}\right)\\
&\leq c_2 M\left(\left(\left(w^{\alpha}\right)^{\#}\left(x\right)\right)^{\delta}\right)+M\left(\left(m_{\lambda}\left(w^{\alpha}\right)\left(x\right)\right)^{\delta}\right)\\
&\leq c_2 M\left(2^{\delta}M\left(w^{\alpha}\right)\left(x\right)^{\delta}\right)+M\left(\left(m_{\lambda}\left(Mw^{\alpha}\right)\left(x\right)\right)^{\delta}\right)\\
&\leq c_3 M\left(M\left(w^{\alpha}\right)\left(x\right)^{\delta}\right)+M\left(\left(m_{\lambda}\left(Mw^{\alpha}\right)\left(x\right)\right)^{\delta}\right)\\
&\leq c_3 M\left(c_4\left(w^{\alpha}\right)\left(x\right)^{\delta}\right)+M\left(\left(c_5 w\left(x\right)^{\alpha}\right)^{\delta}\right)\\
&\leq c_6 Mw\left(x\right)+c_7 Mw\left(x\right)=c_8 Mw\left(x\right)\leq Cw\left(x\right).
\end{aligned}$$

Thus we obtain:

$$w\left(x\right)\leq c_1^{\delta}\left(\left(w^{\alpha}\right)^{\#}\left(x\right)\right)^{\delta}+\left(m_{\lambda}\left(w^{\alpha}\right)\left(x\right)\right)^{\delta}\leq Cw\left(x\right)$$

and then $k\left(x\right)=\frac{w(x)}{c_2\left(\left(w^{\alpha}(x)\right)^{\#}\right)^{\delta}+\left(m_{\lambda}w^{\alpha}(x)\right)^{\delta}}$ satisfies that $k\in L^{\infty}$ and $k^{-1}\in L^{\infty}$.

So $w\left(x\right)=k\left(x\right)\left(C_1\left(\left(w^{\alpha}\left(x\right)\right)^{\#}\right)^{\delta}+C_2\left(m_{\lambda}w^{\alpha}\left(x\right)\right)^{\delta}\right)$ with $k,k^{-1}\in L^{\infty}$ and $\delta\in(0,1)$ for $C_1=c_2$ and $C_2=1$. □

On the other hand we have:

Lemma 28. *If $0<\delta<1$ and $u\in A_1$, then $\left(m_{\lambda}u\left(x\right)\right)^{\delta}\in A_1$.*

Proof. Using that $u\in A_1$, then $Mu\in A_1$ and $m_{\lambda}\left(Mu\right)\approx M\left(Mu\right)\approx Mu\approx u$ and that $\left(MMu\right)^{\delta}\in A_1$ (by Coifman-Rochberg theorem) we have the following inequalities with multiplicative constants that we will renumber:

$$\begin{aligned} M\left((m_\lambda u)^\delta\right) &\leq M\left((m_\lambda Mu)^\delta\right) \\ &\leq M\left((C_1 MMu)^\delta\right) \\ &= C_2 M\left((MMu)^\delta\right) \\ &\leq C_3\,(MMu)^\delta \\ &\leq C_4\,(m_\lambda\,(Mu))^\delta \\ &\leq C_5\,(m_\lambda\,(C_4 u))^\delta \\ &= C_6\,(m_\lambda u)^\delta \end{aligned}$$

and then we get that $(m_\lambda u)^\delta \in A_1$. □

Remark 29. *It's elementary that if v_1, v_2 are non-negative functions with $v_1, v_2 \in A_1$ and if c and d are non-negative constants then $cv_1 + dv_2 \in A_1$ and $[cv_1 + dv_2]_{A_1} \leq \max\{[v_1]_{A_1}, [v_2]_{A_2}\}$.*

Compiling the last two lemmas, the proposition, and the previous remark, we have obtained a result analogous to the Coifman-Rochberg result: Theorem 5, whose utterance we repeat:

Theorem (Theorem 5)**.**

(1) If $0 < \delta < 1$, $f \in L^1_{loc}(\mathbb{R}^n)$ and $u \in A_1$ and C_1, C_2 non-negative constants then $C_1\left(f^\#(x)\right)^\delta + C_2\,(m_\lambda u\,(x))^\delta \in A_1$.

(2) Conversely, if $w \in A_1$ then there are $f \in L^1_{loc}(\mathbb{R}^n)$, $u \in A_1$, non-negative constants C_1 and C_2, and $k\,(x)$with $k, k^{-1} \in L^\infty$ such that $w\,(x) = k\,(x)\left(C_1 f^\#(x)^\delta + C_2 m_\lambda u\,(x)^\delta\right)$.

Proof. The first statement is a consequence of the latter remark and the lemmas telling us that $f^\#(x)^\delta$and $(m_\lambda u\,(x))^\delta$ are in A_1 for $f \in L^1_{loc}$ and $u \in A_1$.

The second was obtained in the latter proposition for $f = u = w^\alpha$ taking a suitable $\alpha > 1$ such that $w^\alpha \in A_1$. The existence of that α is guaranteed by property E. □

Application 30. *As another application of the results we have that for those weights u such that $Mu \in A_\infty$ and hence $Mu \in A_1$ we can improve some known inequalities for singular integral operators. For instance, if T is a*

Calderón-Zygmund singular integral operator (see [8] for a definition) the following weighted inequalities were proved for $1 < p < \infty$ *by C. Pérez ([15]). Previously, J.M. Wilson obtained the first inequality for* $1 < p < 2$*:*

$$\int_{\mathbb{R}^n} |Tf|^p \, u \leq C_p \int_{\mathbb{R}^n} |f|^p \, M^{P+1} u,$$

and then

$$u\left(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}\right) \leq \frac{C_p}{\lambda^p} \int_{\mathbb{R}^n} |f|^p \, M^{P+1} u,$$

and the last one for the case $p = 1$

$$u\left(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}\right) \leq \frac{C_2}{\lambda} \int_{\mathbb{R}^n} |f| \, M^2 u,$$

where P *is the integer part of* p *and* M^k *is the* k*-th iterate composition of* M*. The strong inequality is sharp in the sense that* $P+1$ *cannot be replaced by* P*, and the weak case is sharp when* p *is not an integer. It is an open question (to our knowledge) if it is possible to replace* M^{P+1} *with* M^P *if* $p \in \mathbb{N}$ *and* M^2 *with* M *in the last inequality.*

Now for a weight u *such that* $Mu \in A_\infty$ *we have that actually* $Mu \in A_1$ *and then there are a constant* $C > 0$ *such that for almost every* $x \in \mathbb{R}^n$: $M^2 u(x) \leq CMu(x)$*. Using that if in almost everywhere* $f(x) \leq g(x)$ *then* $Mf(x) \leq Mg(x)$*, we can iterate in* $M^2 u(x) \leq CMu(x)$ *to obtain* $M^k u(x) \leq C^k Mu(x)$*, then with* $C = C_p^k$ *we have for the Calderón-Zygmund singular integral operators and the weights* u *with* $Mu \in A_\infty$*:*

$$\int_{\mathbb{R}^n} |Tf|^p \, u \leq C \int_{\mathbb{R}^n} |f|^p \, Mu$$

$$u\left(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}\right) \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f|^p \, Mu$$

for any $1 < p < \infty$*.*

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