REVERSE HÖLDER INEQUALITES REVISITED: INTERPOLATION, EXTRAPOLATION, INDICES AND DOUBLING

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ABSTRACT. Extending results in [50] and [49] we characterize the classical classes of weights that satisfy reverse Hölder inequalities in terms of indices of suitable families of $K-$functionals of the weights. In particular, we introduce a Samko type of index (cf. [41]) for families of functions, that is based on quasi-monotonicity, and use it to provide an index characterization of the RH_p classes, as well as the limiting class $RH = RH_{LLogL} =. \bigcup_{p>1} RH_p$ (cf. [8]), which in the abstract case involves extrapolation spaces. Reverse Hölder inequalities associated to $L(p,q)$ norms, and non-doubling measures are also treated.

CONTENTS

This paper is in final form and no version of it will be submitted for publication elsewhere.

1. Introduction

The usual applications of interpolation theory deal with the study of scales of function spaces, and the operators acting on them. Indeed, the impact of interpolation theory in classical analysis, pde's, approximation theory, and functional analysis, is well documented (cf. [12], [13], [16], [18], [44], [59], [71], [72], and the references therein). Somewhat less known is the fact that some of the underlying techniques of interpolation theory can be also applied successfully to study problems that one usually does not describe as "interpolation theoretic problems".

In this vein, in [50], [8], [49], [47], [9], [48], [53], we developed new methods to study classes of weights that satisfy reverse Hölder inequalities, using tools from real interpolation theory. It was shown how to transform the classical definitions of the theory of reverse Hölder inequalities into inequalities for suitable families of the $K-$functionals of the weights, that when combined with the basic properties of the theory of real interpolation spaces, like the representation of interpolation norms as averages of end-point norms ("reiteration"), with their crucial "scaling", implied differential inequalities whose solutions yield classical "open properties"[1] like Gehring's Lemma.

To better explain the contents of this paper it will be useful to review some of the basic ideas connecting reverse Hölder inequalities, indices, interpolation, and extrapolation[2]. We refer to Section 2 for unexplained notation and background on interpolation theory and reverse Hölder inequalities.

Given $1 < p < \infty$, we shall say that a weight[3] w satisfies a $p-$reverse Hölder inequality, and we write $w \in RH_p$, if there exists a constant[4] $C > 0$ such that, for all cubes[5] Q, we have

$$(1.1) \qquad \left\{ \frac{1}{|Q|} \int_Q w(x)^p dx \right\}^{1/p} \leq C \left\{ \frac{1}{|Q|} \int_Q w(x)dx \right\}.$$

Fix a cube Q_0. Through the use of local maximal inequalities, the fact that $L^p = (L^1, L^\infty)_{1/p',p}$, known computations of the corresponding $K-$functionals (cf. (2.11)), and the scaling provided by Holmstedt's reiteration formula (cf. [13, Corollary 3.6.2 (b), page 53]),

$$(1.2) \qquad K(t^{1/p}, w; (L^1, L^\infty)_{1/p',p}, L^\infty) \approx \left\{ \int_0^t [K(s, ; L^1, L^\infty)s^{-1/p'}]^p \frac{ds}{s} \right\}^{1/p},$$

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[1]In a somewhat different direction, in [45] other classical "open" or self improving properties, e.g. the open mapping theorem, were connected to a suitable notion of distance for interpolation spaces. The precise relationship between [45], and the developments presented in this paper remains to be investigated.

[2]We have tried to accomodate prospective readers that could be interested in the theory of weighted norm inequalities or interpolation theory but perhaps are not familiar with both areas simultaneously. This has led to a longer introduction, which we hope will facilitate to introduce the underlying ideas to readers that feel that they do not have the adequate background.

[3]A positive locally integrable function on $\mathbb{R}^n$.

[4]usually denoted by $\|w\|_{RH_p}$ (cf. (2.2) below).

[5]In this paper all the cubes are assumed to have their sides parallel to the coordinate axes.

one can see (cf. [50]) that (1.1) implies that with a constant independent of Q_0 we have that for all $0 < t < |Q_0|$,

$$(1.3) \qquad K(t^{1/p}, w\chi_{Q_0}; (L^1(Q_0), L^\infty(Q_0))_{1/p',p}, L^\infty(Q_0))$$
$$\leq Ct^{-1/p'} K(t, w\chi_{Q_0}; L^1(Q_0), L^\infty(Q_0)).$$

Conversely, if there exists a constant $C > 0$ such that (1.3) holds for all cubes then it follows that $w \in RH_p$ (cf. Theorem 1 in Section 3). Moreover, underlying the discussion above is the characterization of RH_p through an implicit differential inequality (cf. [50], [8], [49]). For a weight $w \in RH_p$ and each cube Q, we let $\phi_{w,Q,1/p'}(s) = K(s, w\chi_Q; L^1(Q), L^\infty(Q))s^{-1/p'}$, then there exists a universal constant $C > 0$, such that for all cubes Q,

$$(1.4) \qquad \int_0^t (\phi_{w,Q,1/p'}(s))^p \frac{ds}{s} \leq C(\phi_{w,Q,1/p'}(t))^p, 0 < t < |Q|.$$

The inequality (1.4) is central to our approach to reverse Hölder inequalities (cf. [50], [8], [49]). Moreover, as it turns out, the characterization of the solutions of inequalities of the form (1.4) is one of the achievements of all the classical theories of indices (cf. [7], [17], [44], [46], [49], [63], and the references therein). Index theory shows that for each **fixed** cube Q_0, we have the equivalence

$$(1.5) \qquad \int_0^t \left(\phi_{w,Q_0,1/p'}(s)\right)^p \frac{ds}{s} \leq C \left(\phi_{w,Q_0,1/p'}(t)\right)^p \Leftrightarrow index(\phi_{w,Q_0,1/p'}) > 0,$$

where "$index(\phi_{w,Q_0,1/p'})$" is a number, that can be defined in different ways (e.g. (e.g. [44], [46], [64])) and whose precise definition is not important right now. However, for our purposes in this paper, we need to extend the equivalence (1.5) in order to deal with **all** the cubes Q_0, with a uniform constant C. In other words, we need to extend the notion of index originally defined on single functions to include families of functions.

In this paper we undertake to formalize some of the connections between interpolation methods and the classical methods to study reverse Hölder inequalities. In particular, we develop a suitable definition of indices[6] for the families of functions that allows us to extend the equivalence (1.5) to the realm of families of functions. We define the index of the family $\{K(\cdot, w\chi_Q; L^1(Q), L^\infty(Q))\}_Q$ and obtain in the process a complete characterization of the reverse Hölder classes of weights in terms of our indices (cf. Theorem 6):

$$(1.6) \qquad RH_p = \{w : ind\{K(\cdot, w\chi_Q; L^1(Q), L^\infty(Q))\}_Q > 1/p'\}.$$

This characterization leads to a simple explanation of the open or self improving properties underlying the theory (e.g. Gehring's Lemma). Indeed, if $w \in RH_p$ then by (1.6) it is possible to select $\varepsilon := \varepsilon(w)$, such that $p_0 = p + \varepsilon$, is such that $ind\{K(\cdot, w\chi_Q; L^1(Q), L^\infty(Q))\}_Q > 1/p_0' > 1/p'$ and therefore, once again by (1.6), it follows that $w \in RH_{p_0}$.

The case $p = 1$ requires a different treatment since in this case the inequality (1.1) is true for all weights. Moreover, the usual form of Holmstedt's formula (1.2) does not hold. On the other hand, if we replace $1/p'$ by 0, then (1.4) still makes

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[6]. Our main inspiration for this came from [50] that shows that solutions of (1.4) are quasi-increasing and the work of Samko and her collaborators (cf. [63], [41, Theorem 3.6], [64]), who among other definitions considers an index based on the notion of quasi-monotonicity.

sense and, indeed, plays a rôle in the characterization of the limiting class RH. It turns out that the description of RH is connected with extrapolation spaces (cf. [8], [35], [4]). We shall now develop this point in some detail.

Let

$$(1.7) \qquad\qquad RH = \bigcup_{p>1} RH_p,$$

then the limiting case of (1.6) can be stated as (cf. Theorem 6 (ii))

$$(1.8) \qquad RH = \{w : \ ind\{K(\cdot, w\chi_Q; L^1(Q), L^\infty(Q))\}_Q > 0\}.$$

The elements of RH can be also characterized explicitly in terms of comparisons of their averages that, in the abstract case, involves the use of extrapolation spaces. Indeed, it turns out that the correct reverse Hölder inequality in the limiting case is to compare the $LLogL$ averages of w with its L^1 averages[7] (cf. [29], [30], [8]). The result can be stated as follows. Let RH_{LLogL} be the class of weights[8] w such that there exists a constant $c > 0$, such that for all cubes Q,

$$(1.9) \qquad\qquad \|w\|_{LLogL(Q, \frac{dx}{|Q|})} \le c \left\{ \frac{1}{|Q|} \int_Q w(x)dx \right\},$$

then (cf. Theorem 6 (iii)),

$$(1.10) \qquad\qquad RH = RH_{LLogL}.$$

In this framework, Gehring's Lemma for RH_{LLogL} (cf. [29], [30], [8]) follows from the fact if $ind\{K(\cdot, w\chi_Q; L^1(Q), L^\infty(Q))\}_Q > 0$, then we can choose $p : p(w) > 1$, such that (cf. [49]) $ind\{K(\cdot, w\chi_Q; L^1(Q), L^\infty(Q))\}_Q > 1/p'$.

It is important to mention that the formalism we have outlined above works, and indeed was first developed, in the general setting of interpolation/extrapolation spaces. In the abstract theory we replace the pair (L^1, L^∞) by a Banach pair (X_0, X_1), and $(L^1, L^\infty)_{1/p',p}$ by $(X_0, X_1)_{\theta,q}$ (cf. Section 2.2, Definition 5) and, of course, there are no considerations of cubes. Note that, in general, the index "q" may not be correlated in a specific way to the first index "θ", and this uncoupling already manifests itself when dealing with $L(p, q)$ spaces, as we now explain.

The Lorentz $L(p, q) = (L^1, L^\infty)_{1/p',q}$ spaces are quintessential interpolation spaces, so it is instructive to indicate some possibly new results on reverse Hölder inequalities for Lorentz spaces, that can be derived using our methods. For this purpose we now recall the appropriate scaling that we use to define averages of Lorentz norms. It will be actually easier to frame the discussion in a slightly more general setting.

Let $X := X(R^n)$ be a rearrangement invariant space, and let X' be the associate space of X. It is well known, and easy to see (cf. [12]), that for every cube Q, we have $\|\chi_Q\|_X \|\chi_Q\|_{X'} = |Q|$, this fact, combined with Hölder's inequality, yields

$$\int |w\chi_Q| \le \|w\chi_Q\|_X \|\chi_Q\|_{X'},$$

yields

$$(1.11) \qquad\qquad \frac{1}{|Q|} \int |w\chi_Q| \le \frac{\|w\chi_Q\|_X}{\|\chi_Q\|_X}.$$

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[7]Interestingly, in [8] we arrived first to this formulation using interpolation/extrapolation. It is one instance where interpolation was used as a discovery tool in classical analysis.

[8]see Definition 2

In this context the natural maximal operator is given by (cf. [10], and the references therein),

$$M_X w(x) = \sup_{Q \ni x} \frac{\|w\chi_Q\|_X}{\|\chi_Q\|_X}.$$

It follows that,

$$(1.12) \qquad\qquad Mw(x) \leq M_X w(x),$$

where $M := M_{L^1}$ is the maximal operator of Hardy-Littlewood. For example, if $X = L^p$, $1 \leq p < \infty$, then $M_{L^p} f = \{M|f|^p\}^{1/p}$. The choice $X = L(p,q)$ is of particular interest, for in this case M_X corresponds to the maximal function introduced by Stein [68], that plays a rôle in the theory of Sobolev spaces and other areas of classical Analysis (cf. [54] for a recent reference).

To reverse (1.11), we introduce the class RH_X of weights w such that there exists a constant $c > 0$, such for all cubes Q, it holds

$$(1.13) \qquad\qquad \frac{\|w\chi_Q\|_X}{\|\chi_Q\|_X} \leq c\frac{1}{|Q|} \int w\chi_Q.$$

In terms of maximal functions, (1.13) implies a reversal of (1.12) which, in terms of rearrangements, is given by:

$$(1.14) \qquad\qquad (M_X w)^*(t) \leq c(Mw)^*(t).$$

The class RH_p corresponds to the choice $X = L^p$. Moreover, note that for $X = L(p,q)$, $\|\chi_Q\|_{L(p,q)} = |Q|^{1/p}$ and, therefore, the scaling of (1.13) leads to the consideration of averages controlled by $\frac{1}{t^{1/p}} K(t^{1/p}, f; L(p,q), L^\infty)$. This scaling is compatible with the general definition of reverse Hölder inequalities we give below (cf. (2.13)) and, therefore, our interpolation machinery can be applied, provided we have the appropriate rearrangement inequalities for the corresponding maximal operators. Such inequalities are available for the $L(p,q)$ spaces, with[9] $1 < p \leq q$. Indeed, in this case, the rearrangement inequalities of [10] give

$$(1.15) \qquad\qquad (M_{L(p,q)} f)^*(t) \geq c\frac{1}{t^{1/p}} K(t^{1/p}, f; L(p,q), L^\infty).$$

It follows that, if $w \in RH_{L(p,q)}$, $1 < p \leq q$, then, by (1.14) and (1.15), w satisfies the $K-$functional inequality (cf. (2.13))

$$\frac{1}{t^{1/p}} K(t^{1/p}, f; L(p,q), L^\infty) \leq C\frac{K(t, f; L^1, L^\infty)}{t}.$$

In other words, w satisfies a version of (2.13), and therefore (via Holmstedt's formula!) we can setup an inequality of the form (1.5) and show a Gehring self improving effect, even though the condition $w \in RH_{L(p,q)}, 1 < p \leq q$, is weaker[10] than $w \in RH_p$. However, an argument with indices shows that the index q is not important here (cf. Remark 4) and, in fact, we have,

$$RH_p = RH_{L(p,q)}, 1 < p \leq q < \infty.$$

We refer to Section 5.1 for a more detailed discussion.

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[9]Note that for $q \leq p$, $RH_{L(p,q)} \subset RH_p$, and therefore, $RH_{L(p,q)}$ inherits the self-improvement property from RH_p.

[10]since, indeed, by definition we have $w \in RH_p \Rightarrow w \in RH_{L(p,q)}$.

The interpolation method can be also implemented when dealing with suitable non-doubling weights that are absolutely continuous with respect to Lebesgue measure (cf. [49], [47] for reverse Hölder inequalities, and also [58], where the corresponding theory of A_p weights for non-doubling measure is treated using classical methods). This example should be of interest to classical analysts, and aficionados of interpolation theory. The issue at hand is that in the non-doubling setting the leftmost inequality in the following chain

$$(Mw)^*(t) \approx w^{**}(t) = \frac{K(t, w; L^1, L^\infty)}{t}$$

does not hold (cf. [3]). Thus, a different mechanism is needed to relate the information on the averages of w, coming from conditions like RH_p, to information about $K-$functionals. The appropriate solution in this case is to dispense with the classical maximal operator altogether and work directly with a different expression of the $K-$functional for the weighted pair $(L_w^p(\mathbb{R}^n), L^\infty(\mathbb{R}^n))$. Such formulae was obtained in [3] (cf. also [47] and Section 4.3 below). We shall now review that part of the story.

For a given sequence of disjoint cubes ("packing") $\pi = \{Q_i\}_{i=1}^{|\pi|}$, with $|\pi| = \#$ cubes in π, we associate a linear operator S_π, defined by
(1.16)

$$S_\pi(f)(x) = \sum_{i=1}^{|\pi|} \left(\frac{1}{w(Q_i)} \int_{Q_i} f(y)w(y)dy \right) \chi_{Q_i}(x), \qquad f \in L_w^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n),$$

and let $\left(F_{|f|^p}\right)_w$ be the maximal operator defined by[11]

(1.17) $$\left(F_{|f|^p}\right)_w (t) = \sup_\pi \left(S_\pi(|f|^p)\right)_w^* (t).$$

Then,

(1.18) $$K(t, f; L_w^p(\mathbb{R}^n), L^\infty(\mathbb{R}^n)) \approx t^{1/p} \left(F_{|f|^p}\right)_w (t).$$

Using this tool we can bypass the use of the classical maximal operator and readily show that the reverse Hölder inequalities can be formulated as $K-$functional inequalities of the form (2.13) (cf. Section 4.3, (4.12)), thus making available the interpolation machinery, including the characterization of these classes of weights via indices (cf. (1.6)).

In Section 5 we consider other applications of our theory. For example, the well known connection between weights that satisfy reverse Hölder inequalities and the A_p weights of Muckenhoupt, one of whose manifestations is given by the equality $A_\infty = RH$, which, combined with (1.8), gives an index characterization of A_∞,

$$A_\infty = \{w : ind\{K(\cdot, w\chi_Q; L^1(Q), L^\infty(Q))\}_Q > 0\}.$$

In Section 5.2 we compare our result with the recent characterization of A_∞ obtained in [2] using different indices and without use of interpolation methods. It is shown in [2] that

$$A_\infty = \{w : \widetilde{ind}(w) < 1\},$$

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[11]Here $*_w$ are rearrangements with respect to the measure $w(x)dx$.

where $\widetilde{ind}(w)$ is an index introduced in [2] independently from the theory of indices or interpolation theory. In Section 5.2, (5.5), we give a direct proof of

$$ind\{K(\cdot, w\chi_Q; L^1(Q), L^\infty(Q))\}_Q > 0 \Leftrightarrow \widetilde{ind}(w) < 1,$$

which clarifies the situation.

As another application of our methods, in Section 5.4 we give a simple proof of the important formula obtained by Stromberg-Wheeden (cf. [24]),

$$A_\infty^p = RH_p.$$

We have also included a small section of problems (cf. Section 6) connected with the topics discussed in the paper.

The table of contents should serve as guide to the contents of the paper. A few words about the bibliography are also in order. Documenting the material discussed in the paper has resulted in a relatively large bibliography but, alas, it was not our intention to compile a comprehensive one. We have not attempted to cover the huge amount of material that falls outside our development in this paper. Moreover, since interpolation methods up to this point have not been mainstream in the theory of weighted norm inequalities, and indeed one of the objectives of this paper is to help to try to reverse this situation, our references tend to exhibit a distinctive vintage character. Therefore, we apologize in advance if your favorite papers are not quoted. We should also call attention to the fact that there is a literature that utilizes some of the underlying technical tools that we use here but implemented using a completely different point of view than ours. In particular, without the use of interpolation theoretical methods we developed. A case in point is our use of rearrangements, a technique that, indeed, goes back to early papers on weighted norm inequalities (cf. [57]) and has been treated extensively by several authors (e.g. the Italian school (e.g. [56]) and others.)

2. BACKGROUND: CLASSES OF WEIGHTS AND INTERPOLATION THEORY

In this section we recall some basic definitions. Our main reference on interpolation theory, function spaces and rearrangements will be [12]. The references for theory of weighted norm inequalities we use are [33], [71], and for Gehring's Lemma we refer to [38].

2.1. **Weights.** We start by recalling the definition of RH_p, the class of weights that satisfy a reverse Hölder inequality.

Definition 1. *Let $1 < p < \infty$. A weight w is a positive locally integrable function defined on R^n. We shall say that a weight w belongs to the reverse Hölder class RH_p, if there exists $C := C(w) > 0$ such that for all cubes $Q \subset R^n$, we have:*

$$(2.1) \qquad \left(\frac{1}{|Q|} \int_Q w(x)^p \, dx \right)^{\frac{1}{p}} \leq C \frac{1}{|Q|} \int_Q w(x) \, dx.$$

We let[12]

$$(2.2) \qquad \|w\|_{RH_p} = \inf\{C : (2.1) \text{ holds}\}.$$

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[12]By abuse of language we use the norm symbol here.

We shall also consider the limit class

$$(2.3) \qquad\qquad RH = \bigcup_{p>1} RH_p.$$

The RH_p classes increase as $p \downarrow 1$, so one can ask whether RH can be described by suitable limiting version of (2.1). In this regard we note that, if we simply let $p = 1$ in (2.1), the resulting condition is satisfied by all weights. It turns out that the correct comparison condition in the limiting case $p = 1$ is to replace the $L^p(Q)$ averages by $LLogL(Q)$ averages.

Definition 2. *(cf. [29], [30], [8]) We shall say that a weight w belongs to the reverse Hölder class RH_{LLogL} if there exists $C := C(w) > 0$ such that, for all cubes Q, we have*

$$(2.4) \qquad\qquad \|w\|_{L(LogL)(Q,\frac{dx}{|Q|})} \le C\frac{1}{|Q|}\int_Q w(x)\,dx,$$

where

$$\|f\|_{L(LogL)(Q,\frac{dx}{|Q|})} = \inf\{r : \frac{1}{|Q|}\int_Q \frac{|f(y)|}{r}\log(e + \frac{|f(y)|}{r})dy \le 1\},$$

and we let

$$(2.5) \qquad\qquad \|w\|_{RH_{LLogL}} = \inf\{C : (2.4) \text{ holds}\}.$$

It is shown in the references mentioned above that (2.4) is, indeed, the correct condition to describe the limiting class of weights that satisfy a reverse Hölder inequality (cf. Section 4, Theorem 2 below):

$$RH = RH_{LLogL}.$$

The reverse Hölder classes are connected with the Muckenhoupt A_p classes of weights. On some occasion we shall refer to the connection between these classes of weights, so we now briefly recall the definitions.

Definition 3. *Let $p \in (1,\infty)$. We shall say that a weight w belongs to the (Muckenhoupt) class A_p, if there exists $C := C(w) > 0$ such that, for all cubes Q, it holds*

$$(2.6) \qquad\qquad \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{\frac{-1}{p-1}}\,dx\right)^{p-1} \le C.$$

We shall say that w belongs to A_1, if there exists $C := C(w) > 0$ such that for every cube Q :

$$(2.7) \qquad\qquad \frac{1}{|Q|}\int_Q w(y)\,dy \le Cw(x),$$

for almost every $x \in Q$. Equivalently, $w \in A_1$ iff there exists $C := C(w) > 0$ such that

$$(2.8) \qquad\qquad Mw(x) \le Cw(x), a.e.,$$

where M is the Hardy-Littlewood maximal operator defined by

$$(2.9) \qquad\qquad Mw(x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |w(s)|\,ds.$$

The corresponding limiting class of weights A_∞ is defined by

$$(2.10) \qquad A_\infty = \bigcup_{p \geq 1} A_p.$$

One basic connection between the RH_p and A_p classes of weights is given by the following well known limiting case identity (cf. [21])

$$RH = A_\infty.$$

Summarizing the results for the limiting classes of weights discussed, we have

$$RH_{LLogL} = RH = A_\infty.$$

The A_p and RH_p classes enjoy the following well known self-improvement property that can be described informally[13] as follows:

$$\text{``}w \in A_p \Rightarrow w \in A_{p-\varepsilon}\text{''} \quad \text{and} \quad \text{``}w \in RH_p \Rightarrow w \in RH_{p+\varepsilon}\text{''},$$

where the "ε" depends on w. In Section 4 we show how the indices introduced in this paper give simple proofs of results in [49], [50] that in particular, show that these "open" or self-improving properties admit a very simple interpretation in the abstract setting of interpolation theory.

2.2. Real interpolation and K-functionals.

Given a compatible pair of Banach spaces[14], $\vec{X} = (X_0, X_1)$, the $K - functional$ of an element $w \in X_0 + X_1$, is the nonnegative concave function on $\mathbb{R}_+$ defined by[15]

$$(2.11) \qquad K\left(t, w; \vec{X}\right) = \inf_{w=x_0+x_1} \left\{ \|x_1\|_{X_0} + t \|x_0\|_{X_1} \right\}, t > 0.$$

The interpolation spaces $\vec{X}_{\theta,q}, \theta \in (0,1), 1 \leq q \leq \infty$, are defined by

$$\vec{X}_{\theta,q} = \{ w \in X_0 + X_1 : \|w\|_{\vec{X}_{\theta,q}} = \left\{ \int_0^\infty [K(s, w; \vec{X}) s^{-\theta}]^q ds \right\}^{1/q} < \infty \},$$

with the usual modification when $q = \infty$.

We shall say that a Banach pair $\vec{X} = (X_0, X_1)$ is "ordered" if $X_1 \subset X_0$, in which case we let $n := n_{X_0, X_1} = \sup_{f \in X_1} \frac{\|f\|_{X_0}}{\|f\|_{X_1}}$, be the norm of the corresponding embedding.

Definition 4. *Let $\vec{X} = (X_0, X_1)$ be an ordered pair. Then we let[16] $\vec{X}_{0,1} = \{x : \|x\|_{\vec{X}_{0,1}} = \int_0^n K(s, x; \vec{X}) \frac{ds}{s} < \infty \}$.*

The $\vec{X}_{0,1}$ spaces appear naturally in extrapolation theory (cf. [35], [4]). Their import for our development here comes from the following

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[13]In fact, the self improvement of these classes is interconnected since, if we let $RH_{p'}(w(x)dx)$ denote the class of weights that belong to $RH_{p'}$ with respect to the measure $w(x)dx$, then, as is well known and easy to see, we have $w \in A_p \Leftrightarrow w^{-1} \in RH_{p'}(w(x)dx)$ (cf. [8] and the references therein).

[14]This means that there exists a topological vector space V, such that X_0, X_1 are continuously embedded in V.

[15]When there cannot be confusion we shall simply write $K(t, w)$.

[16]Note that for an ordered pair $\vec{X}$, the $K-$functional is constant for $t > n$. Also note that in general $\int_0^\infty K(s, f; \vec{X}) \frac{ds}{s} < \infty$ implies $f = 0$. For more on this we refer to [35], [4].

Example 1. *(cf. [35]) Let Q be a cube on R^n, and let $\vec{X} = (L^1(Q), L^\infty(Q))$. Then, $(L^1(Q), L^\infty(Q))$ is an ordered pair and*

$$(2.12) \qquad \vec{X}_{0,1}(Q) = (L^1(Q), L^\infty(Q))_{0,1} = \{f : \int_0^{|Q|} f^*(s) \log \frac{|Q|}{s} ds < \infty\}.$$

Proof. Since $\|f\|_{L^1(Q)} \leq |Q| \|f\|_{L^\infty(Q)}$, $n = |Q|$, and $K(t, f; L^1(Q), L^\infty(Q)) = \int_0^t f^*(s) ds = t f^{**}(t)$ becomes constant when $t > |Q|$. Integration by parts yields

$$\int_0^n K(s, f; L^1(Q), L^\infty(Q)) \frac{ds}{s} = \int_0^{|Q|} s f^{**}(s) \frac{ds}{s}$$
$$= \int_0^{|Q|} f^*(s) \log \frac{|Q|}{s} ds,$$

as we wished to show. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

In this abstract context we define "reverse Hölder classes" as follows

Definition 5. *(cf. [50]) Let $\theta \in (0, 1), 1 \leq q < \infty$. Given a Banach pair $\vec{X}$, we let $RH_{\theta,q}(\vec{X})$ be the class of elements $w \in X_0 + X_1$ such that there exists a constant $C = C_w(\vec{X}) > 0$, such that*

$$(2.13) \qquad K(t, w; \vec{X}_{\theta,q}, X_1) \leq Ct \frac{K(t^{\frac{1}{1-\theta}}, w; \vec{X})}{t^{\frac{1}{1-\theta}}}, \text{ for all } t > 0.$$

We let

$$(2.14) \qquad\qquad \|w\|_{RH_{\theta,q}(\vec{X})} = \inf\{C : (2.13) \text{ holds}\}.$$

Moreover, we let

$$(2.15) \qquad\qquad RH(\vec{X}) := \bigcup_{(\theta,q) \in (0,1) \times [1,\infty)} RH_{\theta,q}(\vec{X}).$$

The corresponding limiting class $RH_{0,1}$ is given by

Definition 6. *(cf. [8]) Let $\vec{X}$ be an ordered pair. We shall say that $w \in X_0$ belongs to the class $RH_{0,1}(\vec{X})$ if there exists $C := C_w(\vec{X}) > 0$, such that, for all $0 < t < n$, it holds*

$$(2.16) \qquad\qquad \int_0^t K(s, w; \vec{X}) \frac{ds}{s} \leq CK(t, w; \vec{X}).$$

We let

$$\|w\|_{RH_{0,1}(\vec{X})} = \inf\{C : (2.16) \text{ holds}\}.$$

Remark 1. *It is of interest to point out the connection of (2.16) with a limiting form of Holmstedt's formulae. In fact, recall that using Holmstedt's formula we can rewrite the inequality defining $RH_{\theta,1}(\vec{X})$,*

$$K(t, w, \vec{X}_{\theta,1}, X_1) \leq Ct \frac{K(t^{\frac{1}{1-\theta}}, w; \vec{X})}{t^{\frac{1}{1-\theta}}},$$

as

$$\int_0^{t^{\frac{1}{1-\theta}}} s^{-\theta} K(s, w; \vec{X}) \frac{ds}{s} \leq Ct \frac{K(t^{\frac{1}{1-\theta}}, w, \vec{X})}{t^{\frac{1}{1-\theta}}}.$$

But since $s^{-\theta}$ decreases, the last inequality implies

$$t^{\frac{-\theta}{1-\theta}} \int_0^{t^{\frac{1}{1-\theta}}} K(s, w; \vec{X}) \frac{ds}{s} \leq Ct \frac{K(t^{\frac{1}{1-\theta}}, w, \vec{X})}{t^{\frac{1}{1-\theta}}}$$

yielding

$$\int_0^{t^{\frac{1}{1-\theta}}} K(s, w; \vec{X}) \frac{ds}{s} \leq CK(t^{\frac{1}{1-\theta}}, w, \vec{X}),$$

therefore if we formally let $\theta = 0$ we obtain (2.16).

The connection between the generalized reverse Hölder inequalities and the classical definitions that were given in Section 2.1 is explained in the next section.

3. Reverse Hölder inequalities: Classical vs Interpolation definitions

In this section we show the precise connection between the class of weights that satisfy the classical reverse Hölder inequalities and the corresponding definitions provided by interpolation theory.

Theorem 1. *(cf. [50]) Let $p > 1$. Then, $w \in RH_p$ if and only for all cubes Q, $w\chi_Q$, the restriction of w to Q, belongs to $RH_{1-1/p,p}(L^1(Q), L^\infty(Q))$, and*

$$\sup_Q \|w\chi_Q\|_{RH_{1-1/p,p}(L^1(Q), L^\infty(Q))} < \infty.$$

Proof. Suppose that $w \in RH_p$, then for all cubes Q,

$$(3.1) \qquad \left(\frac{1}{|Q|} \int_Q w(x)^p \, dx \right)^{\frac{1}{p}} \leq \|w\|_{RH_p} \frac{1}{|Q|} \int_Q w(x) \, dx.$$

Fix a cube Q_0. Then, for all $x \in Q_0$ we have the pointwise inequality,

$$M_{p,Q_0}(w\chi_{Q_0})(x) = \sup_{Q\ni x, Q\subset Q_0} \left(\frac{1}{|Q|} \int_Q (w\chi_{Q_0}(x))^p dx \right)^{\frac{1}{p}}$$

$$\leq \|w\|_{RH_p} \sup_{Q\ni x, Q\subset Q_0} \frac{1}{|Q|} \int_Q w\chi_{Q_0}(x) dx$$

$$= \|w\|_{RH_p} M_{Q_0}(w\chi_{Q_0})(x).$$

By the well known Herz rearrangement inequalities (cf. [12, Theorem 3.8, pag 122]) applied to M_{Q_0}, we have that, for $0 < t < |Q_0|$, and with absolute constants independent of w, and Q_0,

$$(M_{p,Q_0}(w\chi_{Q_0}))^*(t) \approx \left\{ \frac{1}{t} \int_0^t (w\chi_{Q_0})^*(s)^p ds \right\}^{1/p},$$

$$(M_{Q_0}(w\chi_{Q_0}))^*(t) \approx \frac{1}{t} \int_0^t (w\chi_{Q_0})^*(s) ds.$$

It follows that there exists a universal constant C, independent of w and Q_0, such that, for all $0 < t < |Q_0|$, we have

$$(3.2) \qquad \left\{ \frac{1}{t} \int_0^t (w\chi_{Q_0})^*(s)^p ds \right\}^{1/p} \leq C \|w\|_{RH_p} \frac{1}{t} \int_0^t (w\chi_{Q_0})^*(s) ds.$$

On the other hand, since $L^p(Q_0) = (L^1(Q_0), L^\infty(Q_0))_{1-1/p,p}$ and the $K-$ functional for the pair $(L^p(Q_0), L^\infty(Q_0)), 1 \le p < \infty$, is given by[17]

$$(3.3) \qquad K(t^{1/p}, w\chi_{Q_0}; L^p(Q_0), L^\infty(Q_0)) \approx \left\{ \int_0^t (w\chi_{Q_0})^*(s)^p ds \right\}^{1/p},$$

we can rewrite (3.2) as follows: for all $0 < t < |Q_0|$, we have

$$K(t^{1/p}, w\chi_{Q_0}; (L^1(Q_0), L^\infty(Q_0))_{1-1/p,p}, L^\infty(Q_0))$$

$$(3.4) \qquad \le \tilde{C} \|w\|_{RH_p} t^{-(1/p-1)} K(t, w\chi_{Q_0}, L^1(Q_0), L^\infty(Q_0)).$$

Moreover, since the cube Q_0 was arbitrary, and the constant in (3.4) does not depend on Q_0, we conclude that $w\chi_Q \in RH_{1-1/p,p}(L^1(Q), L^\infty(Q))$ for all cubes Q and, moreover,

$$\sup_Q \|w\chi_Q\|_{RH_{1-1/p,p}(L^1(Q),L^\infty(Q))} \le \tilde{C} \|w\|_{RH_p},$$

as we wished to show.

Conversely, suppose that $\sup_Q \|w\chi_Q\|_{RH_{1-1/p,p}(L^1(Q),L^\infty(Q))} < \infty$. Fix a cube Q_0. Then, for all $t > 0$,

$$K(t^p, w\chi_{Q_0}; (L^1(Q_0), L^\infty(Q_0))_{1-1/p,p}, L^\infty(Q_0))$$
$$\le \left(\sup_Q \|w\chi_Q\|_{RH_{1-1/p,p}(L^1(Q),L^\infty(Q))} \right) t^{-(1/p-1)} K(t, w\chi_{Q_0}, L^1(Q_0), L^\infty(Q_0)).$$

Now, let $t = |Q_0|$ and use the identification (3.3) to obtain that for some absolute constant $\tilde{C}$ not depending on Q_0, it holds

$$\left\{ \frac{1}{|Q_0|} \int_0^{|Q_0|} (w\chi_{Q_0})^*(s)^p ds \right\}^{1/p}$$
$$\le \tilde{C} \left(\sup_Q \|w\chi_Q\|_{RH_{1-1/p,p}(L^1(Q),L^\infty(Q))} \right) \frac{1}{|Q_0|} \int_0^{|Q_0|} (w\chi_{Q_0})^*(s)ds.$$

Whence,

$$(3.5) \qquad \left(\frac{1}{|Q_0|} \int_{Q_0} w(x)^p dx \right)^{\frac{1}{p}}$$
$$\le \tilde{C} \left(\sup_Q \|w\chi_Q\|_{RH_{1-1/p,p}(L^1(Q),L^\infty(Q))} \right) \frac{1}{|Q_0|} \int_{Q_0} w(x) dx.$$

Consequently, since Q_0 was arbitrary,

$$\|w\|_{RH_p} \le \tilde{C} \left(\sup_Q \|w\chi_Q\|_{RH_{1-1/p,p}(L^1(Q),L^\infty(Q))} \right)$$

as we wished to show. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 2. *It follows from the proof that, with constants possibly depending on* $1 < p < \infty$, *we have*

$$(3.6) \qquad \sup_Q \|w\chi_Q\|_{RH_{1-1/p,p}(L^1(Q),L^\infty(Q))} \approx \|w\|_{RH_p}.$$

---

[17]The equivalence holds with constants independent of $w\chi_{Q_0}$.

3.1. **The limiting case** $p = 1$**.** In order to extend the results of the previous section to the limiting case $p = 1$, and relate $RH_{0,1}$ to the condition provided by Definition 2, we shall need to compare different norms for the space $LLogL$. While such norm comparison results are part of the folklore, it is hard to find references that provide a complete treatment that serves our requirements, therefore, for the sake of completeness, we chose to provide full details in the next lemma,

Lemma 1. *Suppose that* $f \in LLogL_{loc}(\mathbb{R}^n)$. *Then,*
  (i) For all cubes Q

$$\frac{1}{|Q|} \int_Q |f(y)| \log(e + \frac{|f(y)|}{\|f\chi_Q\|_{L^1(Q, \frac{dx}{|Q|})}}) dy \leq 2 \|f\|_{LLogL(Q, \frac{dx}{|Q|})}$$

$$\leq \frac{1}{|Q|} \int_Q |f(y)| \log(e + \frac{|f(y)|}{\|f\chi_Q\|_{L^1(Q, \frac{dx}{|Q|})}}) dy,$$

where $\|f\|_{L(LogL(Q, \frac{dx}{|Q|})}$ *denotes the* $LLogL(Q, \frac{dx}{|Q|})$ *Luxemburg norm of* f,

$$(3.7) \qquad \|f\|_{L(LogL(Q, \frac{dx}{|Q|}))} = \inf\{r : \frac{1}{|Q|} \int_Q \frac{|f(y)|}{r} \log(e + \frac{|f(y)|}{r}) dy \leq 1\}.$$

  (ii) There exists an absolute constant such that for all cubes Q

$$\frac{1}{|Q|} \int_Q |f(y)| \log(e + \frac{|f(y)|}{\|f\chi_Q\|_{L^1(Q, \frac{dx}{|Q|})}}) dy \leq \frac{1}{|Q|} \int_0^{|Q|} (f\chi_Q)^*(s) \log(e + \frac{|Q|}{s}) ds$$

$$\leq c \|f\chi_Q\|_{LLogL(Q, \frac{dx}{|Q|})}.$$

Proof. **(i)** Since the Young's function $y \log(e + y)$ satisfies the Δ_2 condition, the infimum in (3.7) is attained, and we have

$$(3.8) \qquad \frac{1}{|Q|} \int_Q |f(y)| \log(e + \frac{|f(y)|}{\|f\chi_Q\|_{LLogL(Q, \frac{dx}{|Q|})}}) dy = \|f\|_{LLogL(Q, \frac{dx}{|Q|})}.$$

In particular, since $\log(e + \frac{|f(y)|}{\|f\chi_Q\|_{L(LogL(Q, \frac{dx}{|Q|})}}) \geq 1$, we recover the well known fact that

$$(3.9) \qquad \|f\chi_Q\|_{L^1(Q, \frac{dx}{|Q|})} = \frac{1}{|Q|} \int_Q |f(y)|\, dy \leq \|f\|_{LLogL(Q, \frac{dx}{|Q|})}.$$

By (3.9), $\frac{\|f\|_{L(LogL(Q, \frac{dx}{|Q|})}}{\|f\chi_Q\|_{L^1(Q, \frac{dx}{|Q|})}} \geq 1$ and therefore we can write,

$$e + \frac{|f(y)|}{\|f\chi_Q\|_{L^1(Q, \frac{dx}{|Q|})}} = e + \frac{|f(y)|}{\|f\chi_Q\|_{L^1(Q, \frac{dx}{|Q|})}} \frac{\|f\chi_Q\|_{LLogL(Q, \frac{dx}{|Q|})}}{\|f\chi_Q\|_{LLogL(Q, \frac{dx}{|Q|})}}$$

$$\leq (e + \frac{|f(y)|}{\|f\chi_Q\|_{LLogL(Q, \frac{dx}{|Q|})}}) \frac{\|f\chi_Q\|_{LLogL(Q, \frac{dx}{|Q|})}}{\|f\chi_Q\|_{L^1(Q, \frac{dx}{|Q|})}}.$$

Consequently,

$$\frac{1}{|Q|}\int_Q |f(y)|\log(e+\frac{|f(y)|}{\|f\chi_Q\|_{L^1(Q,\frac{dx}{|Q|})}})dy$$

$$\leq \frac{1}{|Q|}\int_Q |f(y)|\log\left\{(e+\frac{|f(y)|}{\|f\|_{LLogL(Q,\frac{dx}{|Q|})}})\frac{\|f\|_{LLogL(Q,\frac{dx}{|Q|})}}{\|f\chi_Q\|_{L^1(Q,\frac{dx}{|Q|})}}\right\}dy$$

$$= (I)+(II),$$

where

$$(I)=\frac{1}{|Q|}\int_Q |f(y)|\log\left(e+\frac{|f(y)|}{\|f\chi_Q\|_{LLogL(Q,\frac{dx}{|Q|})}}\right)dy=\|f\chi_Q\|_{LLogL(Q,\frac{dx}{|Q|})} \text{ (by (3.8))}$$

$$(II)=\log\left(\frac{\|f\chi_Q\|_{LLogL(Q,\frac{dx}{|Q|})}}{\|f\chi_Q\|_{L^1(Q,\frac{dx}{|Q|})}}\right)\frac{1}{|Q|}\int_Q |f(y)|\,dy\leq\|f\chi_Q\|_{LLogL(Q,\frac{dx}{|Q|})}.$$

Therefore, we have shown that

$$\frac{1}{|Q|}\int_Q |f(y)|\log(e+\frac{|f(y)|}{\|f\chi_Q\|_{L^1(Q,\frac{dx}{|Q|})}})dy\leq 2\|f\chi_Q\|_{LLogL(Q,\frac{dx}{|Q|})}.$$

On the other hand, using successively (3.8) and (3.9), we obtain

$$\|f\|_{LLogL(Q,\frac{dx}{|Q|})}=\frac{1}{|Q|}\int_Q |f(y)|\log(e+\frac{|f(y)|}{\|f\|_{LLogL(Q,\frac{dx}{|Q|})}})dy$$

$$\leq\frac{1}{|Q|}\int_Q |f(y)|\log\left(e+\frac{|f(y)|}{\|f\chi_Q\|_{L^1(Q,\frac{dx}{|Q|})}}\right)dy.$$

(ii) By the definition of rearrangement,

(3.10)
$$\frac{1}{|Q|}\int_Q |f(y)|\log\left(e+\frac{|f(y)|}{\|f\chi_Q\|_{L^1(Q,\frac{dx}{|Q|})}}\right)dy=$$

$$\frac{1}{|Q|}\int_0^{|Q|}(f\chi_Q)^*(s)\log\left(e+\frac{(f\chi_Q)^*(s)}{\|f\chi_Q\|_{L^1(Q,\frac{dx}{|Q|})}}\right)dy.$$

Now, since $(f\chi_Q)^*(u)$ is decreasing, we have that, for all $0<s<|Q|$,

$$(f\chi_Q)^*(s)\leq\frac{1}{s}\int_0^{|Q|}(f\chi_Q)^*(u)du$$

$$=\frac{1}{s}|Q|\,\|f\chi_Q\|_{L^1(Q,\frac{dx}{|Q|})}.$$

Inserting this information in (3.10) we see that

$$\frac{1}{|Q|}\int_Q |f(y)|\log(e+\frac{|f(y)|}{\|f\chi_Q\|_{L^1(Q,\frac{dx}{|Q|})}})dy\leq\frac{1}{|Q|}\int_0^{|Q|}(f\chi_Q)^*(s)\log(e+\frac{|Q|}{s})ds.$$

Let $\Omega = \{s \in (0, |Q|) : \left(\frac{e|Q|}{s}\right)^{1/2} \leq \frac{(f\chi_Q)^*(s)}{\|f\|_{LLogL(Q, \frac{dx}{|Q|})}}\}$, then we see that, with absolute constants, we have

$$\frac{1}{|Q|} \int_0^{|Q|} (f\chi_Q)^*(s) \log(e + \frac{|Q|}{s}) ds \approx \frac{c}{|Q|} \int_0^{|Q|} (f\chi_Q)^*(s) \log(\frac{e|Q|}{s}) ds$$

$$= \frac{c}{|Q|} \int_\Omega (f\chi_Q)^*(s) \log(\frac{e|Q|}{s}) ds$$

$$+ \frac{c}{|Q|} \int_{(0.|Q|)\setminus\Omega} (f\chi_Q)^*(s) \log(\frac{e|Q|}{s}) ds$$

$$= (I) + (II).$$

To estimate (I) we proceed as follows[18]

$$(I) \leq \frac{c}{|Q|} \int_\Omega (f\chi_Q)^*(s) \log(\frac{(f\chi_Q)^*(s)}{\|f\chi_Q\|_{LLogL(Q, \frac{dx}{|Q|})}}) ds$$

$$\leq \frac{c}{|Q|} \int_0^{|Q|} (f\chi_Q)^*(s) \log(\frac{(f\chi_Q)^*(s)}{\|f\chi_Q\|_{LLogL(Q, \frac{dx}{|Q|})}}) ds$$

$$= \frac{c}{|Q|} \int_Q |f(y)| \log(\frac{|f(y)|}{\|f\chi_Q\|_{LLogL(Q, \frac{dx}{|Q|})}}) dy$$

$$\leq \frac{c}{|Q|} \int_Q |f(y)| \log(e + \frac{|f(y)|}{\|f\chi_Q\|_{LLogL(Q, \frac{dx}{|Q|})}}) dy$$

$$= c\|f\chi_Q\|_{LLogL(Q, \frac{dx}{|Q|})}.$$

Likewise,

$$(II) = \frac{c}{|Q|} \|f\chi_Q\|_{LLogL(Q, \frac{dx}{|Q|})} \int_{(0.|Q|)\setminus\Omega} \frac{(f\chi_Q)^*(s)}{\|f\chi_Q\|_{LLogL(Q, \frac{dx}{|Q|})}} \log(\frac{e|Q|}{s}) ds$$

$$\leq \frac{c}{|Q|} \|f\chi_Q\|_{LLogL(Q, \frac{dx}{|Q|})} \int_0^{|Q|} \left(\frac{e|Q|}{s}\right)^{1/2} \log(\frac{e|Q|}{s}) ds$$

$$= c\|f\chi_Q\|_{LLogL(Q, \frac{dx}{|Q|})} \int_0^1 \left(\frac{e}{u}\right)^{1/2} \log(\frac{e}{u}) du$$

$$= \tilde{c}\|f\chi_Q\|_{LLogL(Q, \frac{dx}{|Q|})}.$$

$\square$

Now we can state the version of Theorem 1 that corresponds to the case $p = 1$.

Theorem 2. $w \in RH_{L(LogL)}$ *if and only for all cubes* $Q, w\chi_Q$, *the restriction of* w *to the cube* Q, *belongs to* $RH_{0,1}(L^1(Q), L^\infty(Q))$ *and*

$$\sup_Q \|w\chi_Q\|_{RH_{0,1}(L^1(Q), L^\infty(Q))} \approx \|w\|_{RH_{LLogL}}.$$

Proof. For a fixed cube Q_0, we let

$$M_{L(LogL),Q_0}(w\chi_{Q_0})(x) = \sup_{x \varepsilon Q \subset Q_0} \|w\chi_{Q_0}\|_{LLogL(\frac{dx}{|Q|})(Q)}$$

---

[18]Where c indicates an absolute constant whose value may change from line to line.

Suppose that $w \in RH_{LLogL}$, then for $x \in Q_0$,

$$M_{L(LogL),Q_0}(w\chi_{Q_0})(x) \leq \|w\|_{RH_{LLogL}} M_{Q_0}(w\chi_{Q_0})(x)$$

Combining the previous estimate with the localized version of Perez's estimate for the iterated maximal operator (cf. [60, (13) page 174])

$$M_{Q_0}(M_{Q_0}(w\chi_{Q_0}))(x) \leq C M_{L(LogL),Q_0}(w\chi_{Q_0})(x)$$

yields

$$(3.11) \qquad M_{Q_0}(M_{Q_0}(w\chi_{Q_0}))(x) \leq C \|w\|_{RH_{LLogL}} M_{Q_0}(w\chi_{Q_0})(x), a.e. \text{ on } Q_0.$$

Taking rearrangements and using Herz's estimate for the maximal function we see that, for $0 < t < |Q_0|$, we have

$$\begin{aligned}
\frac{1}{t}\int_0^t (w\chi_{Q_0})^{**}(s)ds &\approx \frac{1}{t}\int_0^t (M_{Q_0}(w\chi_{Q_0}))^*(s)ds \\
&\approx (M_{Q_0}(M_{Q_0}(w\chi_{Q_0})))^*(t) \\
&\leq C \|w\|_{RH_{LLogL}} (M_{Q_0}(w\chi_{Q_0}))^*(t) \\
&\approx C \|w\|_{RH_{LLogL}} \frac{1}{t}\int_0^t (w\chi_{Q_0})^*(s)ds.
\end{aligned}$$

In terms of $K-$functionals we therefore have that, for $0 < t < |Q_0|$,
(3.12)
$$\int_0^t K(s, w\chi_{Q_0}; L^1(Q_0), L^\infty(Q_0))\frac{ds}{s} \leq C \|w\|_{RH_{LLogL}} K(t, w\chi_{Q_0}; L^1(Q_0), L^\infty(Q_0)),$$

where C is a universal constant. It follows from (2.16)), that for all cubes $Q, w\chi_Q \in RH_{0,1}(L^1(Q), L^\infty(Q))$, and, moreover,

$$\sup_Q \|w\chi_Q\|_{RH_{0,1}(L^1(Q), L^\infty(Q))} \leq C \|w\|_{RH_{LLogL}}.$$

Conversely, suppose that for all cubes Q, $w\chi_Q \in RH_{0,1}(L^1(Q), L^\infty(Q))$, with $\sup_Q \|w\chi_Q\|_{RH_{0,1}(L^1(Q),L^\infty(Q))} < \infty$. Therefore, by (2.16), for any cube Q_0, it holds

$$\int_0^t K(s, w\chi_{Q_0}; L^1(Q_0), L^\infty(Q_0))\frac{ds}{s} \leq$$
$$C \left(\sup_Q \|w\chi_Q\|_{RH_{0,1}(L^1(Q),L^\infty(Q))}\right) K(t, w\chi_{Q_0}; L^1(Q_0), L^\infty(Q_0)).$$

Let $t = |Q_0|$. Then, using Example 1, we obtain

$$\frac{1}{|Q_0|}\int_0^{|Q|} (w\chi_{Q_0})^*(s)\log\frac{|Q_0|}{s}ds \leq$$
$$(3.13) \qquad C \left(\sup_Q \|w\chi_Q\|_{RH_{0,1}(L^1(Q),L^\infty(Q))}\right)\frac{\|w\chi_{Q_0}\|_{L^1(Q_0)}}{|Q_0|}.$$

Combining with Lemma 1 it follows, that for all cubes Q_0,

$$\|w\chi_{Q_0}\|_{LLogL(\frac{dx}{|Q_0|}, Q_0)} \leq C \left(\sup_Q \|w\chi_Q\|_{RH_{0,1}(L^1(Q),L^\infty(Q))}\right)\frac{\|w\chi_{Q_0}\|_{L^1(Q_0)}}{|Q_0|}.$$

Consequently, $w \in RH_{LLogL}$, and

$$\|w\|_{RH_{LLogL}} \leq C \left(\sup_Q \|w\chi_Q\|_{RH_{0,1}(L^1(Q),L^\infty(Q))} \right),$$

as we wished to show. $\qquad\square$

4. Reverse Hölder classes and Indices

Interpolation theory reduces some of the basic issues around reverse Hölder inequalities to the control of simple integrals. Although the results in this section can be easily extended to a more general context we will focus our development on the specific needs of this paper. So we shall consider families of functions indexed by cubes that are constructed as follows. For each cube Q we associate a function ϕ_Q defined on $(0, |Q|)$. We assume that the functions ϕ_Q are continuous, increasing and such that $\phi_Q(s)/s$ decreases. Let $\beta \in [0,1)$, $q \geq 1$, we let $\phi_{Q,\beta}(s) = s^{-\beta}\phi_Q(s)$, and $\phi_{Q,\beta,q}(s) = [s^{-\beta}\phi_Q(s)]^q$; in particular, $\phi_{Q,0,1}(s) = \phi_Q(s)$, and $\phi_{Q,\beta,1}(s) = \phi_{Q,\beta}(s)$. The prototype examples are constructed using the functions $\phi_Q(s)$ of the form $\phi_{w,Q}(s) = K(s, w\chi_Q, L^1(Q), L^\infty(Q))$, and their multiparameter versions $\phi_{w,Q,\beta,q}(s) = [s^{-\beta}\phi_{w,Q}(s)]^q$, where w is a given weight.

The elementary techniques we use to estimate the integrals involving such functions are displayed in the next Lemma. We note parenthetically (cf. part (iii) of Lemma 2) that the properties of the functions allow us to achieve "global control" from "local control".

Our development in this section builds extensively on the work of Samko and her collaborators (cf. [63], [41], [64], and the references therein) although the specific results dealing with families of functions are apparently new.

We start with a definition:

Definition 7. *A non-negative function ϕ on an interval $(0, l) \subset \mathbb{R}$ is said to be almost increasing (a.i.) if there is a constant $C \geq 1$ (the constant of almost increase) such that $\phi(s) \leq C\phi(t)$ for all $s \leq t$ with $s, t \in (0, l)$.*

Now, we will present a lemma that will play a crucial rôle in what follows.

Lemma 2. *Let w be a weight and let $q \geq 1, \beta \in [0,1)$. The following conditions are equivalent:*

(i) There exists a constant $C > 0$ such that for all cubes Q,

$$(4.1) \qquad \int_0^t \phi_{w,Q,\beta,q}(s)\frac{ds}{s} \leq C\phi_{w,Q,\beta,q}(t), \text{ for all } t \in (0, |Q|).$$

(ii) There exists $\delta > 0$ such that for all cubes Q, $\phi_{w,Q,\beta,q}(s)s^{-\delta}$ is a.i. on $(0, |Q|)$, with constant of almost increase independent of Q and β.

(iii) There exists $\delta > 0, \gamma \in (0,1)$ such that for all cubes Q, $\phi_{w,Q,\beta,q}(s)s^{-\delta}$ is a.i. on $(0, \gamma|Q|)$, with constant of almost increase independent of Q and β.

Proof. $(i) \implies (ii)$. This is an elementary differential inequality argument (e.g. cf. [50]) which we include for the sake of completeness. Let

$$F_{w,Q,\beta,q}(t) = \int_0^t \phi_{w,Q,\beta,q}(s)\frac{ds}{s}.$$

Then (i) can be rewritten as

$$F_{w,Q,\beta,q}(t) \leq Ct(F_{w,Q,\beta,q}(t))'.$$

Therefore,

$$\left(\frac{1}{C}\ln t\right)' \leq (\ln F_{w,Q,\beta,q}(t))',$$

so that for $0 < x < y < |Q|$, we have

$$\ln\left(\frac{y}{x}\right)^{1/C} \leq \ln\frac{F_{w,Q,\beta,q}(y)}{F_{w,Q,\beta,q}(x)},$$

yielding,

$$x^{-1/C}F_{w,Q,\beta,q}(x) \leq y^{-1/C}F_{w,Q,\beta,q}(y) \leq y^{-1/C}C\phi_{w,Q,\beta,q}(y).$$

Combining the last inequality with,

$$\begin{aligned}
F_{w,Q,\beta,q}(x) &= \int_0^x s^{q(1-\beta)}\left[\frac{\phi_{w,Q}(s)}{s}\right]^q\frac{ds}{s} \\
&\geq \left(\frac{\phi_{w,Q}(x)}{x}\right)^q\frac{x^{q(1-\beta)}}{q(1-\beta)} \\
&= \frac{\phi_{w,Q,\beta,q}(x)}{q(1-\beta)},
\end{aligned}$$

implies

$$\begin{aligned}
x^{-1/C}\phi_{w,Q,\beta,q}(x) &\leq (1-\beta)qCy^{-1/C}\phi_{w,Q,\beta,q}(y) \\
&\leq qCy^{-1/C}\phi_{Q,\beta,q}(y).
\end{aligned}$$

$(ii) \Rightarrow (iii)$. Is trivial.

$(iii) \Rightarrow (i)$. Suppose that there exists $\delta > 0, \gamma \in (0,1)$ such that for all Q, $\phi_{w,Q,\beta,q}(s)s^{-\delta}$ is almost increasing on $(0, \gamma|Q|)$, with constant of a.i. C, independent of Q and β. Consider two cases. If $t < \gamma|Q|$, then we can write

$$\begin{aligned}
\int_0^t \phi_{w,Q,\beta,q}(s)\frac{ds}{s} &= \int_0^t \phi_{w,Q,\beta,q}(s)s^{-\delta}s^\delta\frac{ds}{s} \\
&\leq C\phi_{w,Q,\beta,q}(t)t^{-\delta}\frac{t^\delta}{\delta} \\
&= \frac{C}{\delta}\phi_{w,Q,\beta,q}(t).
\end{aligned}$$

Now suppose that $t \in (\gamma|Q|, |Q|)$. Then

$$\begin{aligned}
\int_0^t \phi_{w,Q,\beta,q}(s)\frac{ds}{s} &= \int_0^{\gamma|Q|} \phi_{w,Q,\beta,q}(s)\frac{ds}{s} + \int_{\gamma|Q|}^t \phi_{w,Q,\beta,q}(s)\frac{ds}{s} \\
&= (I) + (II).
\end{aligned}$$

By the first part of the proof

$$
\begin{aligned}
(I) &= \int_0^{\gamma|Q|} \phi_{w,Q,\beta,q}(s)\frac{ds}{s} \\
&\leq \frac{C}{\delta}\phi_{w,Q,\beta,q}(\gamma|Q|) \\
&= \frac{C}{\delta}\left(\phi_{w,Q}(\gamma|Q|)\right)^q \gamma^{-q\beta}|Q|^{-q\beta} \\
&\leq \frac{C}{\delta}\left(\phi_{w,Q}(t)\right)^q \gamma^{-\beta q}t^{-q\beta} \qquad (\text{since } \phi_{w,Q} \text{ increases and } t < |Q|) \\
&= \frac{C}{\delta}\gamma^{-\beta q}\phi_{w,Q,\beta,q}(t).
\end{aligned}
$$

To estimate the remaining integral we use successively that $\phi_{w,Q}$ increases and $t < |Q|$, to obtain

$$
\begin{aligned}
(II) &= \int_{\gamma|Q|}^t \phi_{w,Q,\beta,q}(s)\frac{ds}{s} \\
&= \int_{\gamma|Q|}^t \left(\phi_{w,Q}(s)\right)^q s^{-\beta q}\frac{ds}{s} \\
&\leq \left(\phi_{w,Q}(t)\right)^q \int_{\gamma|Q|}^t s^{-\beta q-1}ds \\
&\leq \left(\phi_{w,Q}(t)\right)^q \frac{1}{\beta q}\frac{t^{\beta q}-(\gamma|Q|)^{\beta q}}{t^{\beta q}(\gamma|Q|)^{\beta q}} \\
&\leq \phi_{w,Q,\beta,q}(t)\frac{1}{\beta q}\frac{1-\gamma^{\beta q}}{\gamma^{\beta q}}.
\end{aligned}
$$

Combining the estimates for (I) and (II) we obtain

$$
\int_0^t \phi_{w,Q,\beta}(s)\frac{ds}{s} \leq \left(\frac{C}{\delta}\gamma^{-\beta q}+\frac{1}{\beta q}\frac{1-\gamma^{\beta q}}{\gamma^{\beta q}}\right)\phi_{w,Q,\beta}(t).
$$

But it is easy to obtain a bound independent of β on the right hand side. Indeed, by elementary calculus we see that the function $f(x) = x\ln(x) - \frac{1}{\beta q}\left(x^{q\beta}-1\right)$ is increasing on $[1,+\infty)$ and $f(1) = 0$, therefore $\gamma^{-1}\ln\left(\gamma^{-1}\right) \geq f(\frac{1}{\gamma}) > \frac{1}{\beta q}\left(\frac{1-\gamma^{\beta q}}{\gamma^{\beta q}}\right)$, while $\frac{C}{\delta}\gamma^{-\beta q} \leq \frac{C}{\delta}\gamma^{-1}$. Therefore, we obtain

$$
\int_0^t \phi_{w,Q,\beta,q}(s)\frac{ds}{s} \leq \left(\frac{C}{\delta}\gamma^{-1}+\gamma^{-1}\ln\left(\gamma^{-1}\right)\right)\phi_{w,Q,\beta,q}(t),
$$

and the desired result follows. $\qquad\qquad\square$

The preceding Lemma combined with the work of Samko and her collaborators (cf. (4.3) below) motivated the following definition

Definition 8. *Let w be a given weight and let $\beta \in [0,1), q \geq 1$. Consider family of functions $\{\phi_{w,Q,\beta,q}\}_Q$ as above. We define the index ind of $\{\phi_{w,Q,\beta,q}\}_Q$ as follows,*

(4.2)

$$ind\{\phi_{w,Q,\beta,q}\}_Q = \sup\{\delta \geq 0 : \exists \gamma \in (0,1) \text{ such that } \text{ for all cubes } Q, \phi_{w,Q,\beta,q}(s)s^{-\delta} \text{ is a.i.}$$
$$\text{on } (0,\gamma|Q|), \text{ with constant of a.i. independent of } Q\}$$

When $\beta = 0, q = 1$, we put $\phi_{w,Q,0,1}(s) := \phi_{w,Q}(s)$; then note that $ind\{\phi_{w,Q}\}_Q = ind\{\phi_{w,Q,0,1}\}_Q$.

The same definition applies when dealing with a single function $\phi_{\beta,q}(s) = \left(s^{-\beta}\phi(s)\right)^q$, where, for the sake of comparison, we assume that ϕ is such that $\phi(s)$ increases and $\phi(s)/s$ decreases on $(0, l)$. For single functions we use the following compatible definition (cf. [41, Theorem 3.6, pag 448])

(4.3) $$i\{\phi_{\beta,q}\} = \sup\{\delta \geq 0 : \ \phi_{\beta,q}(s)s^{-\delta} \text{ is a.i. on } (0,l)\}.$$

The following remark will be useful in what follows

Remark 3. *Let w be a weight and let $q \geq 1, \beta \in [0,1)$. Then,*

$$ind\{\phi_{w,Q,\beta}\}_Q > 0 \Leftrightarrow ind\{\phi_{w,Q,\beta,q}\}_Q > 0.$$

Likewise,

$$i\{\phi_{\beta,q}\} > 0 \Leftrightarrow i\{\phi_\beta\} > 0.$$

Proof. The result follows directly from Definitions 8 and (4.3). For example, note that if $\phi_{Q,\beta}(s)s^{-\delta}$ is a.i. then $(\phi_{Q,\beta}(s))^q s^{-\tilde{\delta}}$ is a.i., with $\tilde{\delta} = \delta q$; and conversely if $(\phi_{Q,\beta}(s))^q s^{-\delta}$ is a.i, then $\phi_{Q,\beta}(s)s^{-\delta/q}$ is a.i.. $\qquad\square$

With this definition we can now reformulate Lemma 2 as follows

Proposition 1. *Let w be a weight, and let $q \geq 1, \beta \in [0,1)$. The following are equivalent:*

(i) There exists $C > 0$ independent of Q and β such that for all

$$\int_0^t \phi_{w,Q,\beta,q}(s)\frac{ds}{s} \leq C\phi_{w,Q,\beta,q}(t), \text{ for all } t \in (0, |Q|).$$

(ii)

(4.4) $$ind\{\phi_{w,Q,\beta,q}\}_Q > 0.$$

Proof. Suppose (i) holds. Then, by Lemma 2 (iii), there exists $\delta > 0$ and $\gamma \in (0,1)$ such that for all Q, $\phi_{w,Q,\beta,q}(s)s^{-\delta}$ is a.i. on $(0, \gamma|Q|)$, with constant of a.i. independent of Q. Therefore, (4.4) holds directly from Definition 8. Likewise, if (4.4) holds then Lemma 2 (iii) holds, and therefore (i) holds. $\qquad\square$

We now show that, in some sense, the computation of $ind\{\phi_{w,Q,\beta}\}_Q$ can be reduced to the computation of $ind\{\phi_{w,Q}\}_Q$

Proposition 2.

$$ind\{\phi_{w,Q,\beta,q}\}_Q > 0 \Leftrightarrow ind\{\phi_{w,Q,\beta}\}_Q > 0 \Leftrightarrow ind\{\phi_{w,Q}\}_Q > \beta.$$

Proof. The first equivalence was proved in Remark 3. We therefore only need to prove the second equivalence. Towards this end let us fix an arbitrary cube Q. The case $\beta = 0$ holds by definition since $\phi_{w,Q,0} = \phi_{w,Q}$. Therefore we shall now assume that $\beta > 0$. Suppose, moreover, that $ind\{\phi_{w,Q}\}_Q > \beta$, then we can find $\delta > 0, \gamma \in (0,1)$, such that $\delta > \beta$ and $\phi_{w,Q}(s)s^{-\delta}$ is a.i. on $(0, \gamma|Q|)$. Therefore, since

$$\phi_{w,Q,\beta}(s)s^{-(\delta-\beta)} = \phi_{w,Q}(s)s^{-\delta}$$

is almost increasing on $(0, \gamma|Q|)$, with $\delta - \beta > 0$, and since Q was arbitrary, we see that $ind\{\phi_{w,Q,\beta}\}_Q > 0$. Conversely, if $ind\{\phi_{w,Q,\beta}\}_Q > 0$, then we can find $\delta > 0$

such that for any cube Q, $\phi_{w,Q,\beta}(s)s^{-\delta} = \phi_{w,Q}(s)s^{-(\delta+\beta)}$ is a.i. on $(0, \gamma |Q|)$ for some fixed $\gamma \in (0,1)$. Therefore, since

$$\phi_{w,Q}(s)s^{-\beta} = \left(\phi_{w,Q}(s)s^{-(\delta+\beta)}\right)s^{\delta}$$

we see that

$$ind\{\phi_{w,Q}\}_Q > \beta.$$

$\square$

The usual definitions of indices in the literature concern the index of one function. We shall now compare the results in this section with classical results using the more common definitions of indices. For comparison purposes[19] we let ϕ be defined on $(0, l)$, with ϕ increasing and $\phi(s)/s$ decreasing. Then many definitions are equivalent. Here we shall specialize our results and consider only functions of the form $\Psi(s) = (s^{-\beta}\phi(s))^q$. Let (cf. [7], [63]),

$$\alpha_\Psi = \sup_{x>1} \frac{\ln\left(\varliminf_{h\longrightarrow 0}\frac{\Psi(xh)}{\Psi(h)}\right)}{\ln x}.$$

Then we have the classical result (cf. [7], [44], [46], [49], [63], and the references therein giving the same result under different definitions of indices)

Lemma 3. *The following are equivalent:*
(i) There exists a constant $C > 0$ such that

$$(4.5) \qquad \int_0^t \Psi(s)\frac{ds}{s} \leq C\Psi(t), \text{ for all } t \in (0, l).$$

(ii) $\alpha_\Psi > 0$.

Combining Proposition 1 and Lemma 3 we see that our definition of index of a single function (4.3) is compatible with the classical ones.

Corollary 1. *Let Ψ be a function defined on $(0, l)$ as above. Then,*

$$\alpha_\Psi > 0 \Leftrightarrow i\{\phi_\Psi\} > 0.$$

Proof. The proof of Proposition (1) for single functions gives us that (4.5) holds if and only if $i\{\phi_\Psi\} > 0$. On the other hand, by Lemma 3 we know that (4.5) holds if and only if $\alpha_\Psi > 0$. The result follows. $\square$

Example 2. *The compatibility of the index (for a single function) with the classical indices is discussed in [41]. In this example we show a simple calculation that hints the reason why the index considered here coincides with classical indices for the classes of functions under consideration. Suppose that $\phi(s)s^{-\delta}$ is almost increasing (a.i.), then, for some constant $c \geq 1$, we have that for $x > 1$,*

$$(xh)^{-\delta}\phi(xh) \geq \frac{1}{c}h^{-\delta}\phi(h).$$

---

[19]The results for the classical indices are valid under less restrictive conditions.

It follows that

$$
\begin{aligned}
\alpha_\phi &= \sup_{x>1} \frac{\ln\left(\varinjlim_{h\longrightarrow 0} \frac{\phi(xh)}{\phi(h)}\right)}{\ln x} \\
&\geq \sup_{x>1} \frac{\ln\left(\frac{1}{c}x^\delta\right)}{\ln x} \\
&= \sup_{x>1}\{\delta + \frac{\ln(\frac{1}{c})}{\ln x}\} \\
&= \delta.
\end{aligned}
$$

4.1. Characterization of abstract reverse Hölder classes via indices.
In this section we essentially show how the results of Mastylo-Milman [49] can be obtained using the indices we have introduced in this paper.

Theorem 3. *Let $\vec{X}$ be a Banach pair, $\theta \in (0,1)$, and $q \geq 1$. Then,*

$$
RH_{\theta,q}(\vec{X}) = \{w \in X_0 + X_1 : i\{K(\cdot,w;\vec{X})\} > \theta\}.
$$

Proof. We shall use the following special case of Holmstedt's formula (cf. [13, Corollary 3.6.2 (b), pag 53]) (with constants dependent on θ, q but not on w)

$$
(4.6)\qquad K(t,w;\vec{X}_{\theta,q},X_1) \approx \left\{\int_0^{t^{1/(1-\theta)}} [s^{-\theta}K(s,w;\vec{X})]^q \frac{ds}{s}\right\}^{1/q}.
$$

Fix $(\theta,q) \in (0,1) \times [1,\infty)$. Let $w \in RH_{\theta,q}(\vec{X})$. Using (4.6) we can rewrite (2.13) as
(4.7)

$$
\left\{\int_0^{t^{1/(1-\theta)}} [s^{-\theta}K(s,w;\vec{X})]^q \frac{ds}{s}\right\}^{1/q} \leq C \|w\|_{RH_{\theta,q}(\vec{X})} \, t\frac{K(t^{\frac{1}{1-\theta}},w,\vec{X})}{t^{\frac{1}{1-\theta}}}, \forall t > 0,
$$

which simplifies to

$$
(4.8)\qquad \int_0^t [s^{-\theta}K(s,w;\vec{X})]^q \frac{ds}{s} \leq C \|w\|_{RH_{\theta,q}(\vec{X})}^q [t^{-\theta}K(t,w,\vec{X})]^q, \forall t > 0.
$$

Since $K(s,w;\vec{X})$ increases and $\frac{K(s,w;\vec{X})}{s}$ decreases, by Lemma 3 and Corollary 1 we have

$$
(4.9)\qquad i\{[(\cdot)^{-\theta}K(\cdot,w;\vec{X})]^q\} > 0.
$$

Consequently, by Proposition 2, it follows that

$$
i\{K(\cdot,w;\vec{X})\} > \theta.
$$

It is easy to see that all the steps can be now reversed. Indeed, if the previous inequality holds, then, by Proposition 2, we see that (4.9) holds and, by Lemma 3, we find that (4.8) holds for all $t > 0$. Changing $t \to t^{1/(1-\theta)}$ in the resulting inequality, we successively see that (4.7), (4.6), and, finally, (2.13) hold, as we wished to show. $\square$

Remark 4. *Note that the second index does not appear in the abstract characterization of $RH_{\theta,q}(\vec{X})$, therefore it follows that, for all $q \geq 1$,*

$$
\begin{aligned}
RH_{\theta,q}(\vec{X}) &= \{w \in X_0 + X_1 : i\{K(\cdot,w;\vec{X})\} > \theta\} \\
&= RH_{\theta,1}(\vec{X}).
\end{aligned}
$$

The previous analysis also yields the following characterization of the limiting class $RH(\vec{X})$ defined by

$$RH(\vec{X}) = \bigcup_{(\theta,q)\in(0,1)\times[1,\infty)} RH_{\theta,q}(\vec{X}).$$

Theorem 4.

$$RH(\vec{X}) = \{w : i\{K(\cdot, w; \vec{X})\} > 0\}.$$

Proof. Let $w \in \bigcup_{(\theta,q)\in(0,1)\times[1,\infty)} RH_{\theta,q}(\vec{X})$. It follows that there exist $\theta \in (0,1)$, and $q \geq 1$, such that $x \in RH_{\theta,q}(\vec{X})$. Therefore, by the previous theorem, $i\{K(\cdot, w; \vec{X})\} > \theta > 0$.

Conversely, suppose that

$$i\{K(\cdot, w; \vec{X})\} > 0.$$

Let $q \geq 1$, and select θ such that $i\{K(\cdot, w; \vec{X})\} > \theta > 0$. Then, by Theorem 3, $w \in RH_{\theta,q}(\vec{X}) \subset RH(\vec{X})$. $\qquad\square$

The limiting case $\theta = 0$ can be obtained using the same arguments.

Corollary 2. *Let $\vec{X}$ be an ordered Banach pair. Then*

$$RH_{0,1}(\vec{X}) = RH(\vec{X}).$$

Proof. Let n be the norm of the embedding, $X_1 \subset X_0$. By definition, $w \in RH_{0,1}(\vec{X})$ if and only for all $0 < t < n$,

$$(4.10) \qquad \int_0^t K(s, w; \vec{X})\frac{ds}{s} \leq cK(t, w; \vec{X}).$$

By Lemma 3 and Corollary 1, (4.10) is equivalent to

$$i\{K(\cdot, w; \vec{X})\} > 0.$$

Consequently, by Theorem 4, $w \in RH(\vec{X})$.

Conversely, if $w \in RH(\vec{X})$, then, by Theorem 4, $i\{K(\cdot, w; \vec{X})\} > 0$, consequently, by Lemma 3, we find that (4.10) holds, whence $w \in RH_{0,1}(\vec{X})$. $\qquad\square$

In this framework Gehring's Lemma is a triviality

Theorem 5. *(Gehring's Lemma) (i) Let $\theta \in (0,1), 1 \leq q < \infty$. Suppose that $w \in RH_{\theta,q}(\vec{X})$, then there exists $\theta' > \theta$, such that, for all $1 < p < \infty$, $w \in RH_{\theta',p}(\vec{X})$.*

(ii) Suppose that $w \in RH_{0,1}(\vec{X})$ then there exists $\theta' > 0$, $1 < p < \infty$, such that $w \in RH_{\theta',p}(\vec{X})$.

Proof. (i) By Theorem 3, $i\{K(\cdot, w; \vec{X})\} > \theta$. Pick $\theta' \in (\theta, i\{K(\cdot, w; \vec{X})\})$ then by Theorem 3, $w \in RH_{\theta',p}(\vec{X})$, for all $p > 1$.

(ii) Follows directly from Corollary 2. $\qquad\square$

4.2. **Classical Reverse Hölder classes and indices.** In this section we characterize the classical classes of weights that satisfy reverse Hölder inequalities. The results are completely analogous to the ones in the previous section but the characterizations are now given in terms of indices of families of K-functionals of the weights involved.

Let w be a weight and let $p \in [1, \infty)$. The family of functions we use here can be defined as follows. Let w be a weight and for each cube Q, let $\phi_{w,Q}(s) = K(s, w\chi_Q, L^1, L^\infty)$; and moreover, let

$$\phi_{w,Q,1/p'}(s) = s^{-1/p'} K(s, w\chi_Q, L^1, L^\infty), 0 < s < |Q|.$$

$$\phi_{w,Q,1/p',q}(s) = \left(s^{-1/p'} K(s, w\chi_Q, L^1, L^\infty)\right)^q, 0 < s < |Q|.$$

Combining the above results with the characterization of classical reverse Hölder inequalities in terms of abstract Hölder classes that were given in Section 3, we obtain

Theorem 6. *(i) Let $p > 1$, then*

$$RH_p = \{w : ind\{\phi_{w,Q,1/p',p}\}_Q > 0\} = \{w : ind\{\phi_{w,Q}\}_Q > 1/p'\}$$

(ii)

$$RH = \{w : ind\{\phi_{w,Q}\}_Q > 0\}$$

(iii)

$$RH = RH_{LLogL}.$$

Proof. **(i)** Suppose that $w \in RH_p$. Then, by Theorem 1 we have that for all cubes Q, $w\chi_Q \in RH_{1/p',p}(L^1(Q), L^\infty(Q))$ and

$$\sup_Q \|w\chi_Q\|_{RH_{1/p',p}(L^1(Q),L^\infty(Q))} \approx \|w\|_{RH_p}.$$

By Theorem 1 there exists a constant $c > 0$, such that for all cubes Q,

$$\int_0^t \phi_{Q,1/p',p}(s)\frac{ds}{s} \le c\|w\|_{RH_p}^p \phi_{Q,1/p',p}(s), \quad 0 < t < |Q|.$$

Consequently, by Proposition 1 followed by Proposition 2,

$$ind\{\phi_{w,Q}\}_Q > 1/p'.$$

Conversely, suppose that w is such that $ind\{\phi_{w,Q,1/p',p}\}_Q > 0$. Then, by Proposition 1, there exists a constant $c > 0$ such that

$$\int_0^t \phi_{Q,1/p',p}(s)\frac{ds}{s} \le c\phi_{Q,1/p',p}(s), \quad 0 < t < |Q|.$$

It follows by Theorem 1 that

$$\sup_Q \|w\chi_Q\|_{RH_{1-1/p,p}(L^1(Q),L^\infty(Q))} \preceq c,$$

and, moreover, $w \in RH_p$, with

$$\|w\|_{RH_p} \preceq c.$$

(ii) Suppose that $w \in RH$, then $w \in RH_p$ for some p. Then, by part (i), $ind\{\phi_{w,Q}\}_Q > 1/p' > 0$. Conversely, if w is a weight such that $ind\{\phi_{w,Q}\}_Q > 0$, then we can select $p > 1$ close enough to 1 so that $\frac{1}{p'} = 1 - \frac{1}{p} < ind\{\phi_{w,Q}\}_Q$. Therefore, by (i), $w \in RH_p \subset RH$.

(iii) We show first the inclusion $RH \subset RH_{LLogL}$. Suppose that $w \in RH$, then there exists $p > 1$ such that $w \in RH_p$. Now, it is easy to verify that for $0 < \alpha < 1$, we have $\log(e + \frac{1}{x}) \preceq x^{-\alpha}$, $x \in (0, 1)$; consequently, by Hölder's inequality, we have

$$\frac{1}{|Q|} \int_0^{|Q|} (w\chi_Q)^*(s) \log(e + \frac{|Q|}{s}) ds \preceq (\frac{1}{|Q|} \int_0^{|Q|} (w\chi_Q)^*(s)^p ds)^{1/p} (\frac{1}{|Q|} \int_0^{|Q|} |Q|^{-\alpha p'} s^{\alpha p'} ds)^{1/p'}$$

$$\preceq (\frac{1}{|Q|} \int_0^{|Q|} (w\chi_Q)^*(s)^p ds)^{1/p}$$

$$\preceq \frac{1}{|Q|} \int_0^{|Q|} (w\chi_Q)^*(s) ds \text{ (since } w \in RH_p\text{)}.$$

Therefore, the conclusion follows from Lemma 1.

We now prove the opposite inclusion. Suppose that $w \in RH_{LLogL}$. Then, by Theorem 2, for all cubes Q we have that $w\chi_Q \in RH_{0,1}\left(L^1(Q), L^\infty(Q)\right)$ and, moreover, $\sup_Q \|w\chi_Q\|_{RH_{0,1}(L^1(Q), L^\infty(Q))} \approx \|w\|_{RH_{LLogL}}$. It follows that

$$\int_0^t K(s, w\chi_Q; L^1(Q), L^\infty(Q)) \frac{ds}{s} \leq \|w\|_{RH_{LLogL}} K(t, w\chi_Q; L^1(Q), L^\infty(Q)).$$

Consequently, by Proposition 1

$$ind\{K(\cdot, w\chi_Q; L^1(Q), L^\infty(Q))\}_Q > 0.$$

$\square$

4.3. **Non-doubling weights.** Let $1 < p < \infty$. For a locally integrable positive function w, we define the class of reverse Hölder weights $RH_p(w)$ simply replacing dx by $w(x)dx$ in the definition of RH_p. Thus, we say that $g \in RH_p(w)$, if there exists $C \geq 1$ such that for every cube Q, with sides parallel to the coordinate axes, we have

$$\left(\frac{1}{w(Q)} \int_Q g(x)^p w(x) dx\right)^{1/p} \leq \frac{C}{w(Q)} \int_Q g(x) w(x) dx,$$

where $w(Q) = \int_Q w(x) dx$. If the measure $\mu := w(x)dx$ satisfies a doubling condition, i.e., if there exists a constant $c > 0$ such that $\mu(B(x, 2r)) \leq c\mu(B(x, r))$, then for the maximal operator M_w

$$M_w g(x) = \sup_{Q \ni x} \frac{1}{w(Q)} \int_Q g(x) w(x) dx$$

we have the equivalence[20]

(4.11) $$(M_w g)_w^*(t) \approx g_w^{**}(t),$$

which in turn gives us

$$(M_w g)_w^*(t) \approx \frac{K(t, g; L_w^1, L_w^\infty)}{t}.$$

Therefore, we can use the analysis of the previous sections *mutatis mutandi*. On the other hand, when dealing with non-doubling measures (4.11) may fail. In particular, the equivalence

$$t g_w^{**}(t) \approx K(t, g; L_w^1, L_w^\infty),$$

---

[20]here the rearrangements f_w^*, f_w^{**}, etc, are with respect to the measure $w(x)dx$

may not hold, so that the implementation of the interpolation method, as discussed in previous sections, requires a different approach. Fortunately, it is possible to find an alternative formula for the $K-$functional that resolves this obstacle, as explained in the Introduction (cf. (1.16), (1.17) and (1.18) for the relevant formulae). In particular, from the definitions given in the Introduction, for each packing π,

$$S_\pi(g)(x) = \sum_{i=1}^{|\pi|} \left(\frac{1}{w(Q_i)} \int_{Q_i} g(y)w(y)dy \right) \chi_{Q_i}(x), \quad g \in L_w^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n),$$

and for $1 \le p < \infty$, we have (cf. [3])

$$K(t,g;L_w^p,L^\infty) \approx t^{1/p} \sup_\pi \left(S_\pi(|g|^p)\right)_w^*(t).$$

The import of this construction is that it allows to translate the information provided by the definition of a reverse Hölder inequality in the language of $K-$functionals, while avoiding the use of the possibly unbounded maximal operator M_w. Indeed, suppose that $g \in RH_p(w(x)dx)$, then, directly from the definitions, we see that

$$t^{1-1/p}K(t^{1/p}, g; L_w^p(\mathbb{R}^n), L^\infty(\mathbb{R}^n)) \le C \|g\|_{RH_p(w(x)dx)} K(t, g; L_w^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n)).$$

Consequently, if we let $\theta = 1 - 1/p$, we get

$$t^\theta K(t^{1-\theta}, g; L_w^p(\mathbb{R}^n), L^\infty(\mathbb{R}^n)) \le C \|g\|_{RH_p(w(x)dx)} K(t, g; L_w^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n))$$

which finally gives

(4.12) $\qquad K(t^{1-\theta}, g; L_w^p(\mathbb{R}^n), L^\infty(\mathbb{R}^n))$

(4.13) $\qquad \le C \|g\|_{RH_p(w(x)dx)} t^{-\frac{\theta}{1-\theta}} K(t^{\frac{1}{1-\theta}}, g; L_w^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n))$

$$\qquad = C \|g\|_{RH_p(w(x)dx)} t \frac{K(t^{\frac{1}{1-\theta}}, g; L_w^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n))}{t^{\frac{1}{1-\theta}}}.$$

Therefore, (2.13) holds and we have: $g \in RH_p(w(x)dx) \Rightarrow g \in RH_{\theta,p}(L_w^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n))$. We can localize this result using the corresponding local formula for the $K-$functional (cf. [47]): for all cubes Q

$$K(t^{1-\theta}, g\chi_Q; L_w^p(Q), L^\infty(Q)) \le C \|g\|_{RH_p(w(x)dx)} t \frac{K(t^{\frac{1}{1-\theta}}, g\chi_Q; L_w^1(Q), L^\infty(Q))}{t^{\frac{1}{1-\theta}}}.$$

Using the Holmstedt's formula we can proceed with the analysis in the weighted case as we did in the unweighted case. In fact, we obtain that if $g \in RH_p(w(x)dx)$ then, for $0 < t < w(Q) = \int_Q w(x)dx$,

$$\int_0^t [K(s, g\chi_Q; L_w^1(Q), L^\infty(Q))s^{-\theta}]^p \frac{ds}{s} \le C[K(s, g\chi_Q; L_w^1(Q), L^\infty(Q))s^{-\theta}]^p.$$

In order to avoid repetitions we shall leave further details for the interested reader and, in particular, refer to [47] where also the $K-$functional for the pair $(L_w^1(Q), L^\infty(Q))$ is computed.

5. APPLICATIONS AND COMPARISON WITH RESULTS IN THE LITERATURE

5.1. **Reverse Hölder inequalities in the setting of Lorentz spaces.** In this section we consider the class of weights $RH_{L(p,q)}$ that satisfy $L(p,q)$ reverse Hölder's inequalities. Our main result here can be summarized as follows

$$RH_{L(p,q)} = RH_p, 1 < p \leq q.$$

Let us recall some definitions.

Definition 9. Let $1 < p < \infty, 1 \leq q < \infty.$ We let $L(p,q) = \{f : \|f\|_{L(p,q)} = \left(\int_0^\infty f^{**}(t)^q t^{\frac{q}{p}} \frac{dt}{t}\right)^{\frac{1}{q}} < \infty\}.$

Definition 10. Let $1 < p < \infty, 1 \leq q < \infty$. We shall say that w satisfies an $L(p,q)$ reverse Hölder Lorentz inequality, and we shall write $w \in RH_{L(p,q)}$, if and only if there exists a constant $C > 0$ such that, for all cubes Q, we have

$$(5.1) \qquad \frac{\|w\chi_Q\|_{L(p,q)}}{|Q|^{1/p}} \leq \frac{C}{|Q|} \int_Q w(x)dx.$$

We let

$$\|w\|_{RH_{L(p,q)}} = \inf\{C : (5.1) \text{ holds}\}.$$

Theorem 7. Let $1 < p < \infty, p \leq q < \infty$. Then,

$$RH_{L(p,q)} = RH_p.$$

Proof. The containment $RH_p \subset RH_{L(p,q)}$ is automatic since $L^p \subset L(p,q)$. Suppose that $w \in RH_{L(p,q)}$. Fix a cube Q_0. Applying (5.1) to $w\chi_{Q_0}$, we get
$$(5.2)$$
$$M_{L(p,q),Q_0}(w\chi_{Q_0})(x) := \sup_{Q_0 \supset Q \ni x} \frac{\|(w\chi_{Q_0}\chi_Q)\|_{L(p,q)}}{|Q|^{1/p}} \leq C \|w\|_{RH_{L(p,q)}} M(w\chi_{Q_0})(x).$$

Then, taking rearrangements in (5.2), and using the familiar Herz inequality to estimate the right hand side, yields

$$(M_{L(p,q),Q_0}(w\chi_{Q_0}))^*(t) \leq \frac{C \|w\|_{RH_{L(p,q)}}}{t} \int_0^t (w\chi_{Q_0})^*(s)ds, 0 < t < |Q_0|.$$

The left hand side can be estimated by the local version of an estimate obtained in [10, Corollary (i), page 69],

$$\frac{1}{t^{1/p}} \int_0^t [(w\chi_{Q_0})^*(s)s^{1/p}]^q \frac{ds}{s} \leq c(M_{L(p,q),Q_0}w(x))^*(t), 0 < t < |Q_0|.$$

Combining these estimates we thus find that there exists a constant $C > 0$, such that for all $t > 0$,

$$(5.3) \qquad \frac{1}{t^{1/p}} \int_0^t (w\chi_{Q_0})^*(s)^q s^{q/p} \frac{ds}{s} \leq C \frac{\|w\|_{RH_{L(p,q)}}}{t} \int_0^t (w\chi_{Q_0})^*(s)ds.$$

Now, recall that (cf. [13], [70])

$$K(t, w; L(p,q), L^\infty) \approx \left\{ \int_0^{t^p} [w^*(s)s^{1/p}]^q \frac{ds}{s} \right\}^{1/q}, K(t, w; L^1, L^\infty) = \int_0^t w^*(s)ds.$$

Let $\theta = 1 - 1/p = 1/p'$, using Holmstedt's formula in a familiar way we can rewrite (5.3) as

$$\frac{1}{t^{1-\theta}} K(t^{1-\theta}, w\chi_{Q_0}; (L^1, L^\infty)_{\theta,q}, L^\infty) \leq C \frac{\|w\|_{RH_{L(p,q)}}}{t} K(t, w\chi_{Q_0}; L^1, L^\infty),$$

therefore, since Q_0 was arbitrary, a simple change of variables shows that w satisfies

(2.13). More precisely, we see that for all cubes Q, $w\chi_Q \in RH_{1/p',q}(L^1(Q), L^\infty(Q))$, with

$$\sup_Q \|w\chi_Q\|_{RH_{1/p',q}(L^1(Q),L^\infty(Q))} \preceq \|w\|_{RH_{L(p,q)}}.$$

Consequently, applying Theorem 1, we conclude that $w \in RH_p$.   □

Corollary 3. *Let $1 < p \leq q < \infty$. Then, $w \in RH_{L(p,q)}$ if and only if there exists a constant $C > 0$ such that, for all cubes Q, $0 < t < |Q|$, we have*

$$\int_0^t [K(s, w\chi_Q; L^1(Q), L^\infty(Q))s^{-1/p'}]^q \frac{ds}{s} \leq C[t^{-1/p'} K(s, w\chi_Q; L^1(Q), L^\infty(Q))]^q.$$

5.2. **Comparisons with recent results on A_∞.** In the recent paper [2], which is apparently independent of the literature on indices or the interpolation methods, as discussed in this paper, the authors defined an index on weights which, among other interesting applications, was used to characterize the Muckenhoupt class of A_∞ weights. The purpose of this section is to compare the results of [2] with ours.

Let us start by recalling the notion of index defined in [2]. We shall say a weight w has finite index, in the sense of [2], if there exist $r \in (0,1)$ and $\lambda, \widetilde{\gamma} > 0$ such that for all cubes Q, $0 < s \leq t < r|Q|$, it holds

(5.4) $$\frac{(w\chi_Q)^*(s)}{(w\chi_Q)^*(t)} \leq \widetilde{\gamma} \left(\frac{s}{t}\right)^{-\lambda}.$$

In [2] the authors then let

$$\widetilde{ind}(w) = \inf\{\lambda : (5.4) \text{ holds}\},$$

and show that

$$A_\infty = \{w : \widetilde{ind}(w) < 1\}.$$

For comparison we note that since

$$A_\infty = RH$$

from Theorem 6 (ii) we obtain

$$A_\infty = \{w : ind\{K(\cdot, w\chi_Q, L^1(Q), L^\infty(Q))\}_Q > 0\}.$$

In this section we compare and clarify these results by means of a direct proof of

Theorem 8.

(5.5) $$ind\{K(\cdot, w\chi_Q, L^1(Q), L^\infty(Q))\}_Q > 0 \Leftrightarrow \widetilde{ind}(w) < 1.$$

Remark 5. *Before going through the proof of (5.5), let us observe that we can rewrite the condition (5.4) as follows: for all cubes Q, $0 < s \leq t < r|Q|$, it holds that*

$$s(w\chi_Q)^*(s) s^{-(1-\lambda)} \leq \widetilde{\gamma} t(w\chi_Q)^*(t) t^{-(1-\lambda)}.$$

In other words, the function $x(w\chi_Q)^*(x)\,x^{-(1-\lambda)}$ is a.i. on $(0, r\,|Q|)$, with constant of a.i. $\widetilde{\gamma}$ independent of Q. Therefore we readily see that

$$\widetilde{ind}(w) = \sup_{\delta > 0}\{\delta > 0 : \exists \widetilde{\gamma} > 0, r \in (0,1) \text{ such that for all cubes } Q, \ x(w\chi_Q)^*(x)\,x^{-\delta}$$

is a.i. on $(0, r\,|Q|)$ with $\widetilde{\gamma}$ constant of a.i.$\}$.

For comparison, in Definition 8 our index, ind, was defined using functions built around the family of functions $\phi_{w,Q}(t) = K(t, w\chi_Q, L^1(Q), L^\infty(Q)) = \int_0^t (w\chi_Q)^*(s)ds$, and we made some (minimal) use of the fact that $\phi_{w,Q}(t)$ increases and $\frac{\phi_{w,Q}(t)}{t}$ decreases. But if we formally apply our definition to the family of functions $\{t\phi'_{w,Q}(t)\}_Q = \{t\,(w\chi_Q)^*(t)\}_Q$ we see that formally we have

$$\widetilde{ind}(w) = ind\{t\phi'_{w,Q}(t)\}_Q\} = ind\{t(w\chi_Q)^*(t)\}_Q.$$

We are now ready for the **Proof of Theorem 8**

Proof. Suppose that $\widetilde{ind}(w) < 1$. Then there exists $\exists \delta > 0, \gamma \in (0,1)$ such that for all cubes, $t\phi'_{w,Q}(t)t^{-\delta}$ is a.i. on $(0, \gamma\,|Q|)$. It follows that for any $0 < t < h < \gamma\,|Q|$, we have

$$
\begin{aligned}
\phi_{w,Q}(t)t^{-\delta} &= t^{-\delta}\int_0^t \phi'_{w,Q}(s)s^{1-\delta}s^{\delta-1}ds \\
&\leq t^{-\delta}\phi'_{w,Q}(t)t^{1-\delta}\frac{t^\delta}{\delta} \\
&\leq \frac{C}{\delta}\phi'_{w,Q}(h)h^{1-\delta} \\
&= \frac{C}{\delta}h^{-\delta}\left(\phi'_{w,Q}(h)h\right) \\
&\leq \frac{C}{\delta}h^{-\delta}\int_0^h \phi'_{w,Q}(r)dr \\
&= \frac{C}{\delta}h^{-\delta}\phi_{w,Q}(h).
\end{aligned}
$$

Therefore for all cubes Q, $t^{-\delta}\phi_{w,Q}(t)$ is a.i. on $(0, \gamma\,|Q|)$. Thus, $ind\{\phi_{w,Q}\}_Q > 0$.

Conversely, if $ind\{\phi_{w,Q}\}_Q > 0$, then by Theorem 6 and Lemma 2 there exists $p > 1$, $C > 0$, such that for $t \in (0, |Q|)$

$$\left\{\frac{1}{t}\int_0^t [(w\chi_Q)^{**}(s)]^p ds\right\}^{1/p} \leq C\left\{\frac{1}{t}\int_0^t (w\chi_Q)^*(s)ds\right\},$$

which implies

$$(5.6) \qquad \left\{\int_0^t [(w\chi_Q)^*(s)]^p ds\right\}^{1/p} \leq C\left\{t^{-1/p'}\int_0^t (w\chi_Q)^*(s)ds\right\}.$$

Let $\rho > 1$ be a number will be chosen precisely later. Then, for $t \in \left(0, \frac{|Q|}{\rho}\right)$, we have

$$
\begin{aligned}
\int_0^t (w\chi_Q)^*(s)ds &\leq \left\{\int_0^t [(w\chi_Q)^*(s)]^p ds\right\}^{1/p} t^{1/p'} \\
&\leq \left\{\int_0^{t\rho} [(w\chi_Q)^*(s)]^p ds\right\}^{1/p} t^{1/p'} \\
&\leq C\left\{(t\rho)^{-1/p'} \int_0^{\rho t} (w\chi_Q)^*(s)ds\right\} t^{1/p'} \text{ (by (5.6))} \\
&= C\rho^{-1/p'} \int_0^t (w\chi_Q)^*(s)ds + C\rho^{-1/p'} \int_t^{\rho t} (w\chi_Q)^*(s)ds
\end{aligned}
$$

Rearranging terms, and using the fact that $(w\chi_Q)^*$ is decreasing, we find

$$
(1 - C\rho^{-1/p'}) \int_0^t (w\chi_Q)^*(s)ds \leq C\rho^{-1/p'}(\rho - 1)t(w\chi_Q)^*(t).
$$

Therefore if we choose $\rho > 1$ such that $C\rho^{-1/p'} < 1$ and use once again the fact that $(w\chi_Q)^*$ is decreasing, we obtain that on $\left(0, \frac{|Q|}{\rho}\right)$,

$$
(5.7) \qquad t(w\chi_Q)^*(t) \leq \int_0^t (w\chi_Q)^*(s)ds \leq \frac{C\rho^{-1/p'}(\rho - 1)}{(1 - C\rho^{-1/p'})} t(w\chi_Q)^*(t).
$$

Now, since $ind\{\phi_{w,Q}\}_Q > 0$, there exists $\delta > 0, \gamma \in (0,1)$ such that $t^{-\delta} \int_0^t (w\chi_Q)^*(s)ds$ is a.i. on $(0, \gamma|Q|)$. Consequently, if we further demand that $\rho > 1/\gamma$, we see that (5.7) implies that $t^{-\delta} t(w\chi_Q)^*(t)$ is a.i. and therefore by Remark 5 we find that

$$
\widetilde{ind}(w) < 1,
$$

as we wished to show. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 6. *In retrospect it is interesting to observe that while the theory of [2] was apparently developed independently from theory of indices, and interpolation theory, one of the first results obtained in [2] is the control of integrals of the form $\int_0^t f(t)\,dt$ by $tf(t)$, where f is decreasing.*

5.3. **An interpolation theorem involving extrapolation spaces: Operators acting on RH weights.** Another interesting way of characterizing A_∞, apparently first given by Fujii (cf. [32], [28] and the references therein), can be stated as follows: w belongs to the A_∞ class if and only if there exists a constant C such that for all cubes Q,

$$
(5.8) \qquad\qquad \int_Q M(w\chi_Q)(x)dx \leq C \int_Q w(x)dx.
$$

We investigate the connection of (5.8) with our own characterization of A_∞ using interpolation. More precisely, in this section we prove an abstract interpolation theorem modelled after a result obtained in [2][21], that when applied to the maximal operator shows that if $w \in RH$ then w satisfies the Fujii condition (5.8).

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[21]On closer examination one can see that the result is closely related to an extrapolation version of a theorem due Zygmund (cf. [4], [35], [40] and the discussion in Remark 8 below).

Let $\vec{X}$ be an ordered Banach pair. We recall the definition of $\|\cdot\|_{RH_{0,1}(\vec{X})}$ that we introduced in Definition 6,

$$(5.9) \qquad \|w\|_{RH_{0,1}(\vec{X})} = \inf\{c : \int_0^t K(s,w;;\vec{X})\frac{ds}{s} \le cK(t,w;\vec{X})\}.$$

Let us also recall the definition of the notion of "generalized weak types $(1,1),(\infty,\infty)$" as given in [26]. We shall say that T is of generalized weak types $(1,1),(\infty,\infty)$, if there exists a constant $C > 0$ such that

$$(5.10) \qquad \frac{K(r,Tf;\vec{X})}{r} \le C\{\frac{1}{r}\int_0^r K(s,f;\vec{X})\frac{ds}{s} + \int_r^\infty \frac{K(s,f;\vec{X})}{s}\frac{ds}{s}\}, r > 0.$$

Theorem 9. *Let $\vec{X}$ be an ordered Banach pair, and let n be the norm of the embedding $X_1 \subset X_0$. Let T be an operator of generalized weak types $(1,1),(\infty,\infty)$. Then, there exists an absolute constant $c > 0$, such that*

$$(5.11) \quad \int_0^t \frac{K(r,Tw;\vec{X})}{r}dr \le c\,\|w\|_{A_0}\,(\|w\|^2_{RH_{0,1}(\vec{X})} + \|w\|_{RH_{0,1}(\vec{X})} + 1), 0 < t < n,$$

$$(5.12) \qquad \|Tf\|_{\vec{X}_{0,1}} \le c\,\|w\|_{A_0}\,(\|w\|^2_{RH_{0,1}(\vec{X})} + 1 + \|w\|_{RH_{0,1}(\vec{X})}).$$

Proof. Let $w \in RH_{0,1}(\vec{X})$. Integrating (5.10) we obtain,

$$\int_0^t \frac{K(r,Tf;\vec{X})}{r}dr \preceq \int_0^t \frac{1}{r}\int_0^r K(s,w;\vec{X})\frac{ds}{s}dr + \int_0^t\int_r^\infty \frac{K(s,w;\vec{X})}{s}\frac{ds}{s}dr$$
$$= (I) + (II).$$

Using (5.9) (twice) we find

$$(I) \preceq \int_0^t \frac{1}{r}\,\|w\|_{RH_{0,1}(\vec{X})}\,K(r,w;\vec{X})dr$$
$$\preceq \|w\|^2_{RH_{0,1}(\vec{X})}\,K(t,w;\vec{X})$$
$$\preceq \|w\|^2_{RH_{0,1}(\vec{X})}\,\|w\|_{A_0}\,.$$

To estimate (II) we integrate by parts. For this purpose note that $K(s,w;\vec{X}) \le \lim_{s\to\infty} K(s,w;\vec{X}) = \|w\|_{A_0}$, and therefore

$$\lim_{r\to 0}(r\int_r^\infty \frac{K(s,w;\vec{X})}{s}\frac{ds}{s}) \le \lim_{r\to 0}(r\,\|w\|_{A_0}\int_r^\infty \frac{ds}{s^2}) = \|w\|_{A_0}\,.$$

Consequently, we get

$$(II) \preceq r\int_r^\infty \frac{K(s,w;\vec{X})}{s}\frac{ds}{s}\Big|_0^t + \int_0^t \frac{K(r,w;\vec{X})}{r}dr$$
$$\preceq t\int_t^\infty \frac{K(s,w;\vec{X})}{s}\frac{ds}{s} + \|w\|_{RH_{0,1}(\vec{X})}\,K(t,w;\vec{X})$$
$$\preceq \|w\|_{A_0} + \|w\|_{RH_{0,1}(\vec{X})}\,\|w\|_{A_0}\,.$$

Combining estimates yields,

$$(5.13) \qquad \int_0^t \frac{K(r,Tw;\vec{X})}{r}dr \preceq \|w\|_{A_0}\,(\|w\|^2_{RH_{0,1}(\vec{X})} + \|w\|_{RH_{0,1}(\vec{X})} + 1).$$

Letting $t \to n$ we then obtain (5.12). $\qquad\qquad\square$

We apply this result to the family of pairs $\vec{X} = (L^1(Q), L^\infty(Q))$, and the maximal operator M. Indeed, as it is well known, the maximal operator M satisfies (5.10). For the benefit of the reader we offer a quick verification here using the familiar Herz's equivalence. Indeed, we have

$$\frac{K(r, Mf, \vec{X})}{r} = (Mf)^{**}(r) = \frac{1}{r}\int_0^r (Mf)^*(s)ds \approx \frac{1}{r}\int_0^r f^{**}(s)ds = \frac{1}{r}\int_0^r K(s, f, \vec{X})\frac{ds}{s}.$$

Suppose now that $w \in RH_{LLogL}$, then, by Theorem 2, we have that, for all cubes Q, $w \in RH_{0,1}(L^1(Q), L^\infty(Q))$, and

$$\sup_Q \|w\|_{RH_{0,1}(L^1(Q), L^\infty(Q))} \approx \|w\|_{RH_{LLogL}} .$$

If we apply (5.13) with $t = |Q|$, we have

$$\int_0^{|Q|} \frac{K(r, M(w\chi_Q); \vec{X})}{r}dr \preceq \|w\|_{L^1(Q)} (\|w\|^2_{RH_{LLogL}} + \|w\|_{RH_{LLogL}} + 1).$$

Now, we observe that

$$\int_Q M(w\chi_Q)(x)dx = \int_0^{|Q|} [M(w\chi_Q)]^*(r)dr$$
$$\leq \int_0^{|Q|} \frac{K(r, M(w\chi_Q); \vec{X})}{r}dr.$$

Combining these estimates we obtain

$$\int_Q M(w\chi_Q)(x)dx \preceq \|w\|_{L^1(Q)} (\|w\|^2_{RH_{LLogL}} + \|w\|_{RH_{LLogL}} + 1)$$

(5.14)
$$= \int_Q w(x)dx(\|w\|^2_{RH_{LLogL}} + \|w\|_{RH_{LLogL}} + 1)$$

This shows that our characterization of $RH_{LLogL} = RH = A_\infty$, implies Fujii's condition.

In the next remark we show directly how the condition (5.8) implies the defining condition of RH_{LLogL}.

Remark 7. *Suppose that w satisfies Fujii's condition (5.8). Let $x \in R^n$, then for any cube $x \in Q$ we have (cf. the argument in [60, pag 174])*

$$\frac{1}{|Q|}\int_Q M(w)(y)dy \leq \frac{1}{|Q|}\int_Q M(w\chi_{3Q})(y)dy + \frac{1}{|Q|}\int_Q M(w(1 - \chi_{3Q}))(y)dy$$
$$\leq \frac{c}{|3Q|}\int_{3Q} M(w\chi_{3Q})(y)dy + c\inf_{z\in Q} Mw(z)$$
$$\leq \frac{\tilde{c}}{|3Q|}\int_{3Q} w(x)dx + cMw(x) \ (by \ (5.8))$$
$$\leq CMw(x).$$

Consequently, for all $x \in R^n$,

$$M(Mw)(x) \leq CMw(x).$$

Taking rearrangements, and then applying Herz's inequality, yields

(5.15)
$$(M(Mw))^*(t) \preceq (Mw)^*(t).$$

Applying Herz's equivalence (repeatedly), and the known calculation of the corresponding $K-$functional, we get the following equivalent expressions to the left and right hand sides of (5.15):

$$(M(Mw))^* (t) \approx \frac{1}{t} \int_0^t (Mw)^*(s)ds \approx \frac{1}{t} \int_0^t f^{**}(s)ds = \frac{1}{t} \int_0^t K(s,f;L^1,L^\infty)\frac{ds}{s},$$

$$(Mw)^* (t) \approx \frac{1}{t} \int_0^t f^*(s)ds = K(t,f;L^1,L^\infty).$$

Rewriting (5.15) using this information we get

$$\int_0^t K(s,f;L^1,L^\infty)\frac{ds}{s} \preceq K(t,f;L^1,L^\infty).$$

Thus, if w is a weight satisfies (5.8) then $w \in RH_{LLogL}$.

Remark 8. *We cannot resist to point out a connection to extrapolation theory. Indeed, the theory of [40] produces many examples of operators weak types $(1,1),(\infty,\infty)$ by means of extrapolating inequalities. A prototype result can be stated as follows: if T is an operator on a real interpolation scale $\{\vec{X}_{\theta,q}\}_{\theta,q}$, $\theta \in (0,1)$, $q \geq 1$, such that*

$$\|T\|_{\vec{X}_{\theta,q}\to\vec{X}_{\theta,q}} \preceq (1-\theta)^{-1}\theta^{-1}$$

then T satisfies (5.8) (cf. [40]). In particular, if $\vec{X}$ is ordered and we are interested only on the behavior on spaces near the larger space X_0, that is when $\theta \to 0$, then operators that satisfy $\|T\|_{\vec{X}_{\theta,q}\to\vec{X}_{\theta,q}} \preceq \theta^{-1}$ as $\theta \to 0$, can be characterized by (cf. [40])

$$(5.16) \qquad \frac{K(r,Tf;\vec{X})}{r} \leq C\{\frac{1}{r} \int_0^r K(s,f;\vec{X})\frac{ds}{s}\}$$

In particular, when $\vec{X} = (L^1,L^\infty)$ this leads to the rearrangement inequalities,

$$(5.17) \qquad \frac{1}{r} \int_0^r (Tf)^*(s)ds \leq \frac{C}{r} \int_0^r f^{**}(s)ds.$$

The idea behind Theorem 9 is that if $f \in RH_{LLogL}$ we have

$$\frac{1}{r} \int_0^r K(s,f;\vec{X})\frac{ds}{s} \leq \|f\|_{RH_{LLogL}} \frac{K(r,f;\vec{X})}{r}$$

therefore if we integrate (5.17) we can use the RH_{LLogL} condition twice on the right hand side to obtain

$$\int_0^t \frac{K(r,Tf;L^1,L^\infty)}{r}dr \leq C \|f\|_{RH_{LLogL}}^2 K(t,f;L^1,L^\infty)$$

which effectively reverses (5.16)!

5.4. **The Stromberg-Wheeden Theorem.** In this section we apply our theory to give a simple proof of the Stromberg-Wheeden theorem, which is arguably one of the cornerstones of the classical theory of weighted norm inequalities (cf. [24]).

Theorem 10. $w \in RH_p$ if and only if $w^p \in A_\infty$

Proof. Suppose that $w^p \in A_\infty$. By the characterization of A_∞ given in 6, $w^p \in A_\infty$ if and only if

(5.18) $$ind\{K(t, w^p\chi_Q; L^1(Q), L^\infty(Q))\}_Q > 0.$$

It follows that there exists $\delta > 0$ such that $\frac{K(t,w^p\chi_Q;L^1(Q),L^\infty(Q))}{t^\delta}$ a.i. on $(0, \gamma|Q|)$, but then the proof of Theorem 8 shows that we $[(w\chi_Q)^*(s)]^p s^{1-\delta}$ is a.i.; therefore, raising this function to the power $1/p$, yields that $(w\chi_Q)^*(s)s^{\frac{1-\delta}{p}}$ is also a.i. Consequently, if we let $\mu = 1 - 1/p + \delta/p$, then we can write $\frac{1-\delta}{p} = 1 - \mu$, and once again by the proof of Theorem 8 we find that $K(s, w\chi_Q; L^1(Q), L^\infty(Q))s^{-\mu} = K(s, w\chi_Q; L^1(Q), L^\infty(Q))s^{-1/p'-\delta/p}$ is a.i., whence

(5.19) $$ind\{K(\cdot, w\chi_Q; L^1(Q), L^\infty(Q))\}_Q > 1/p'.$$

Consequently, by Theorem 6

$$w \in RH_p.$$

Conversely, all the steps can be reversed. Indeed, suppose that $w \in RH_p$. Then (5.19) holds, and consequently for some $\delta > 0$, $K(s, w\chi_Q; L^1(Q), L^\infty(Q)))s^{-1/p'-\delta/p}$ is a.i. Now the proof of Theorem 8 implies that $(w\chi_Q)^*(s)s^{\frac{1-\delta}{p}}$ is a.i. and hence $[(w\chi_Q)^*(s)]^p s^{1-\delta}$ is a.i., and once again by the proof of Theorem 8, $\frac{K(t,w^p\chi_Q;L^1(Q),L^\infty(Q))}{t^\delta}$ is a.i. and therefore (5.18) holds, yielding that $w^p \in A_\infty$. $\qquad\square$

6. Some Problems

We would like to close this paper with some open-ended problems connected with the developments in this paper that we consider of some potential interest. The problems are thus mainly focussed on exploring the connections between weighted norm inequalities and interpolation/extrapolation theoretical methods.

(1) **Interpolation/Extrapolation Methods**: So far, the interpolation methods[22] we have been developing to study classes of weights are built on the real method of interpolation. It is likely that other methods of interpolation could also be of interest in this area. In particular, the rôle of the complex method of interpolation of Calderón ought to be explored. For example, the Calderón method of interpolation of lattices, e.g. the "$X_0^{1-\theta}X_1^\theta$" method, is likely to be relevant in connection with factorizations of weighted norm inequalities and of their underlying classes of weights. Also intriguing are the possible connections with the interpolation method of [25], which allows to treat the real and complex methods of interpolation in a unified way. In particular, [25] introduced a new variant of the $K-$functional that makes this tool available for the complex method of interpolation. We think it could be of interest to explore its application within the framework developed in this paper. Likewise, the method of orbits (cf. [59]) could also play a rôle. In fact, some results that connect the method of orbits and the abstract Gehring Lemma was started to be explored in [9], and there is a detailed application of orbital methods to the study of self-improving (or "open" properties) in [45] (cf. also item **9** below). It seems to us that the theory of indices is likely to have an impact reformulating and clarifying results of [45] and its applications to the theory of weighted norm inequalities

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[22]as opposed to the more common application of interpolation "Then by interpolation".

and pde's. Likewise, the results of Section 5.3, and, in particular, Remark 8, suggest new potential applications of extrapolation theory to weighted norm inequalities.

2 **Function Spaces**: In connection with the set of problems described in **1** above, it would be of interest to study classes of weights that are naturally associated with more general function spaces. In this direction, applications to weighted norm inequalities in the setting of Orlicz spaces would be of obvious interest and a number of results in this direction already appear in [8], [36], [49], [55] and the references therein. In this connection see also item 5 below.

3 **Other classes of weights:** Interpolation methods have the potential to be useful to study other classes of weights. Among the classes of weights awaiting interpolation treatment: The class of A_p weights (cf. [33]), the classes C_p, B_p of weights (cf. [11], [27], [31] and the references therein). Similar questions for two weight type inequalities (cf. [1]).

4 **The role of constants**: in the theory of RH_{LLogL} weights is discussed, for example, in [56], [15]. We ask for a treatment of the role of constants in the theory of weights in the context of the interpolation/extrapolation methods. See also the next item.

5 **Reverse Hardy Inequalities.** There is an interesting connection between Gehring's Lemma and sharp constants for reverse Hardy inequalities (cf. [9], [49], [52]), which we did not discuss in this paper in order not to exasperate the editors of this volume. In this connection it would be interesting to extend the sharp reversed Hardy inequalities for decreasing functions, which are known for L^p norms, to more general function spaces.

6 **The class** $A_{p,q}$ of weights for which the maximal operator of Hardy-Littlewood is bounded on $L(p,q)$ were studied in [20]. In particular, it was shown there that $A_{p,q} = A_p, 1 < p < \infty, 1 < q < \infty$. The case $q = 1$ is discussed in [37]. We are not aware of a systematic study.

7 **The class** RH_∞ was apparently first systematically studied in [24], where it was defined through the use of the minimal operator

$$\mathfrak{M}f(x) = \inf_{Q \ni x} \frac{1}{|Q|} \int_Q |f(x)| \, dx.$$

We say that a weight $w \in RH_\infty$ if there exists $C > 0$ such that $w(x) \leq C\mathfrak{M}w(x)$ $a.e.$ It would be interesting to understand the connection of this operator with interpolation theory.

8 **Discrete Gehring type inequalities via interpolation.** We would like to suggest the project of understanding recent results on discrete reverse Hölder inequalities (cf. [62], see also [61] for related work) using the methods developed in this paper. Likewise, another potential application of our theory is the setting of **metric spaces** (cf. [55] and the references therein).

9 **Self-improving inequalities and PDEs.** This topic is of course of central interest and continues to be a source of problems and inspiration for applications of interpolation methods. Here is a far from complete sample of references in this direction that we happen to be aware of: [5], [6], [14], [19], [30], [42], [66], [67], [69], and the references therein.

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