Linear Algebra and its Applications 614 (2021) 111–124

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On the spectral radius of block graphs with prescribed independence number α [☆]

Cristian M. Conde [a,b,c,*], Ezequiel Dratman [a,c,*], Luciano N. Grippo [a,c,*]

[a] *Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina*
[b] *Instituto Argentino de Matemática Alberto Calderón, Argentina*
[c] *Instituto de Ciencias, Universidad Nacional de General Sarmiento, Argentina*

A R T I C L E   I N F O

Article history:
Received 19 November 2019
Accepted 6 February 2020
Available online 7 February 2020
Submitted by V. Trevisan

MSC:
05 C50
15 A18

Keywords:
Block graphs
Independence number
Spectral radius of a graph

A B S T R A C T

Let $\mathcal{G}(n, \alpha)$ be the class of block graphs on n vertices and prescribed independence number α. In this article we prove that the maximum spectral radius $\rho(G)$, among all graphs $G \in \mathcal{G}(n, \alpha)$, is reached at a unique graph. As a byproduct we obtain an upper for $\rho(G)$, when $G \in \mathcal{G}(n, \alpha)$.

☆ Cristian M. Conde acknowledges partial support from ANPCyT PICT 2017-2522. Ezequiel Dratman and Luciano N. Grippo acknowledge partial support from ANPCyT PICT 2017-1315.

* Corresponding authors.
E-mail addresses: cconde@campus.ungs.edu.ar (C.M. Conde), edratman@campus.ungs.edu.ar (E. Dratman), lgrippo@campus.ungs.edu.ar (L.N. Grippo).

https://doi.org/10.1016/j.laa.2020.02.003
0024-3795/

1. Introduction

To find lower and upper bounds for the spectral radius of a graph is a problem that have attracted the attention of many researchers. Probably, one of the most important motivations for studying this topic is due to a problem posted by Brualdi and Solheid in [1]. They proposed, in that article, to characterize the graphs having the maximum spectral radius among graphs on n vertices in a determined class of graphs. Since then, a wide variety of results on this topic have been published. In addition, finding bounds for the spectral radius of any graph in terms of nonspectral parameters is interesting enough. Many works can be found in the specialized literature. A recently published book summarizes most of the results related to this topic [2].

Lu and Lin find the only graph which maximizes the spectral radius among trees with prescribed independence number [3]. In [4], the authors find the unique connected graph on n vertices, with given connectivity and prescribed independence number having maximal spectral radius. Our contribution, in that line of work, is to find the unique graph $G \in \mathcal{G}(n, \alpha)$ with maximal spectral radius, where $\mathcal{G}(n, \alpha)$ denote the class of block graphs, which are precisely those connected graph whose blocks are complete graphs. Indeed, we have been able to prove that the pineapple on n vertices having independence number α is that unique graph. Notice that this block graphs is a superclass of trees. More precisely, trees are block graphs having all their blocks of cardinality two. In our result this restriction, imposed in [3] on the considered class of graphs, is dropped.

This article is organized as follows. Section 2 is devoted to introduce some preliminary results and definitions. In Section 3 we find the unique block graph on n vertices and given independence number α with maximum spectral radius and we also present an upper bound for the spectral radius of this graph in terms of n and α.

2. Preliminaries

2.1. Definitions

All graphs, mentioned in this article, are finite, have no loops and multiple edges. Let G be a graph. We use $V(G)$ and $E(G)$ to denote the set of vertices and the set of edges of G, respectively. Let v be a vertex of G, $N_G(v)$ (resp. $N_G[v]$) stands for the neighborhood of v (resp. $N_G(v) \cup \{v\}$), if the context is clear the subscript G will be omitted. We use $d_G(v)$ to denote the degree of v in G, or $d(v)$ provided the context is clear. A vertex of degree $|V(G)| - 1$ is called *universal vertex*. By $\overline{G}$ we denote the complement graph of G. Given a set F of edges of G (resp. of $\overline{G}$), we denote by $G - F$ (resp. $G + F$) the graph obtained from G by removing (resp. adding) all the edges in F. If $F = \{e\}$ we use $G - e$ (resp. $G + e$) for short. Let $X \subseteq V(G)$, we use $G[X]$ to denote the graph induced by X. By $G - X$ we denote the graph $G[V(G) \setminus X]$. If $X = \{v\}$ we use $G - v$ for short.

Let $A, B \subseteq V(G)$ we said that A is *complete to* (resp. *anticomplete to*) B if every vertex in A is adjacent (resp. nonadjacent) to every vertex of B. A set of pairwise nonadjacent vertices of G is called an *independent set* (or *stable set*). The *independence number* of G, denoted $\alpha(G)$, is the maximum cardinality of an independent set of G. A *clique* is a set of pairwise adjacent vertices. A *simplicial* vertex of a graph G is a vertex v such that $N(v)$ is a clique. We denote by K_n the complete graph on n vertices. A *tree* is a connected and acyclic graph. By $K_{1,n-1}$ we denote the tree on n vertices having a universal vertex. A *leaf* of a tree is a vertex of degree one and a *support vertex* in a tree is the only vertex adjacent to a leaf. Given two graphs G and H, we use $G = H$ to denote that G and H are isomorphic graphs.

Let G be a graph. We denote by $A(G)$ the adjacency matrix of G, and $\rho(G)$ stands for the spectral radius of $A(G)$, we refer to $\rho(G)$ as the spectral radius of G. Perron-Frobenius theorem implies that the principal eigenvector of $A(G)$ has all its entries either positive or negative. In addition, $\rho(G)$ coincides with the maximum eigenvalue of G. The reader is referred to [5, Ch. 6] for a simple proof of this observation. If x is the principal eigenvector of $A(G)$ which is clearly indexed by $V(G)$, we use x_u to denote the coordinate of x corresponding to the vertex u.

2.2. Technical results

Adding edges to a graph increases the spectral radius of a graph.

Lemma 1. *If G is a graph such that $uv \notin E(G)$, then $\rho(G) < \rho(G + uv)$.*

The problem of finding those graphs that maximizes the spectral radius of a graph on n vertices within a given class $\mathcal{H}$ of graphs, have been solved by means of graphs transformations that increases the spectral radius. We refer to the reader to [2] for more details about this and other techniques. Notice that if $\mathcal{H}$ contains the complete graphs, then K_n maximizes $\rho(G)$ for every $G \in \mathcal{H}$, because of Lemma 1. Lovász and Pelikán in [6] prove that the unique graph with maximum spectral radius among the trees on n vertices is the star $K_{1,n-1}$ defining a partial order within the trees by means of their characteristic polynomials.

Theorem 1. *[6] If T is a tree on n vertices, then $\rho(T) \leq \sqrt{n-1}$. In addition, the equality holds if and only if $T = K_{1,n-1}$.*

Nevertheless, in order to easily prove this result, using the technique of graph transformations, the following result can be used.

Lemma 2. *[7] Let G be a connected graph and let $u, v \in V(G)$ such that $x_u \leq x_v$. If $\{v_1, \dots, v_r\} \subseteq N(u) \setminus N(v)$, then*

$$\rho(G) < \rho(G - \{uv_1, \dots, uv_r\} + \{vv_1, \dots, vv_r\}).$$

Lemma 2 is proved by the first time in [7] but for an easy proof the reader is referred to [8]. We would like to point out that Theorem 1 can be proved, using Lemma 2 by showing that if T is a tree on n vertices having the maximum spectral radius then there is only one support vertex. Otherwise there would exist two support vertices u and v in T satisfying $x_u \leq x_v$ and thus if w is a leaf adjacent to u, and nonadjacent to w, then $\rho(T) < \rho(T - uw + vw)$. Therefore, the tree having the maximum spectral radius is $K_{1,n-1}$ whose only support vertex is its vertex of degree $n-1$.

In the following lemma we consider a set of vertices $u_1, \ldots, u_\ell$ of a graph G, where x_i stands for x_{u_i} for every $1 \leq i \leq \ell$.

Lemma 3. *Let G be a connected graph and let $u_1, \ldots, u_k, u_{k+1}, \ldots, u_\ell \in V(G)$ such that $\sum_{i=1}^{k} x_i \leq \sum_{i=k+1}^{\ell} x_i$, and let $W \subseteq V(G) \setminus \{u_1, \ldots, u_\ell\}$. If $\{u_1, \ldots, u_k\}$ is complete to W and $\{u_{k+1}, \ldots, u_\ell\}$ is anticomplete to W, then*

$$\rho(G) < \rho(G - \{wu_i : w \in W \text{ and } 1 \leq i \leq k\} + \{wu_i : w \in W \text{ and } k+1 \leq i \leq \ell\}).$$

It is worth mentioning that Lemma 3 was presented in [4] by Lu and Lin but in an slightly different way. They prove that $\rho(G) \leq \rho(G^*)$ when $|W| = 1$ and that the inequality is strict when $\sum_{i=1}^{k} x_i < \sum_{i=k+1}^{\ell} x_i$.

3. Block graphs with given independence number

The adjacency matrix of block graph were studied by Bapat and Roy in [9]. Throughout of this section we will need some definitions and concepts introduced next. A vertex v of a graph G is a *cut vertex* if $G - v$ has a number of connected components greater than the number of connected components of G. We use $\mathcal{S}_1(G)$ to denote the set of simplicial vertices of G.

Let H be a graph. A *block* of H, also known as 2-*connected component*, is a maximal connected subgraph of H having no cut vertex. A *block graph* is a connected graph whose blocks are complete graphs. We use $\mathcal{G}(n, \alpha)$ to denote the family of block graphs on n vertices and independence number α. A *simplicial block* of H is a block B having at least a simplicial vertex in H. Let G be a block graph, a *leaf block* is a block of G such that contains exactly one cut vertex of G. Notice that every vertex but one, in a leaf block, is a simplicial vertex.

Next we will state and prove technical results needed to demonstrate the main statement of this section.

Lemma 4. *Let G be a block graph and let B a simplicial block of G. If S is a maximum independent set of G, then $|S \cap V(B)| = 1$. In addition, such a maximum independent set S can be chosen so that $S \cap V(B) = \{v\}$, where v is any simplicial vertex of G such that $N_G[v] = V(B)$.*

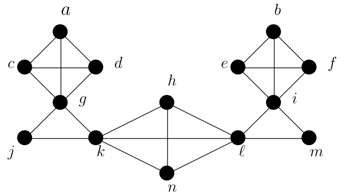

Fig. 1. A block graph, each of its blocks is a simplicial one and its has two leaf blocks.

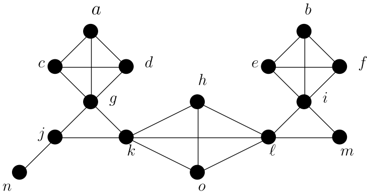

Fig. 2. Block graphs with three leaf blocks, two nonleaf simplicial blocks and a nonsimplicial block.

Proof. Consider a maximum independent set S and let B be a simplicial block of G. Hence there exists a vertex $v \in \mathcal{S}_1(G)$ such that $V(B) = N_G[v]$. Thus there exists a vertex $w \in V(B) \cap S$, because of the maximality of S, since otherwise $S \cup \{v\}$ would be an independent set. Therefore, $|V(B) \cap S| = 1$. In addition, since S is a maximum independent set, $S' = S \setminus \{w\} \cup \{v\}$ is also a maximum independent set. We can proceed in this way with each simplicial block in order to obtain a maximum independent set as stated in the lemma. □

Corollary 1. *Let G be a block graph. Then, there exists a maximum independent set S such that $S \cap V(B) = \{v\}$ for each leaf block B of G, where $N_G[v] = V(B)$.*

Proof. It suffices to notice that if B is a leaf block of G, then B is a simplicial block of G. Therefore, the result immediately follows from Lemma 4. □

Let G be a block graph. We use $\mathcal{L}(G)$ to denote the set of vertices of G belonging to any leaf block. Let B be a block of G. We use $L(B)$ to denote the set of simplicial vertices of those leaf blocks of G having exactly one vertex in common with $V(B)$. By $\ell_G(B)$ we denote the number of these leaf blocks. When the context is clear enough we use $\ell(B)$ for short. In the graph depicted in Fig. 1, if B is the block induced by $\{g, j, k\}$, then $L(B) = \{a, c, d\}$ and $\ell(B) = 1$; and in the graph depicted in Fig. 2, if B is the block induced by $\{g, j, k\}$, then $L(B) = \{a, c, d, n\}$ and $\ell(B) = 2$.

Corollary 2. *If G is a block graph and B is a leaf block of G, then*

$$\alpha(G) = \alpha(G - B) + 1.$$

Proof. The proof follows from Lemma 4. □

Lemma 5. *Let G be a block graph and let B be a leaf block of $H = G - (\mathcal{L}(G) \cap \mathcal{S}_1(G))$, where v is its only cut vertex of H in $V(B)$. Then, the following conditions hold:*

1. $\alpha(G) = \alpha(G - (V(B) \cup L(B))) + \ell(B) + 1$, *if $V(B)$ has a simplicial vertex in G.*
2. $\alpha(G) = \alpha(G - ((V(B) \cup L(B)) \setminus \{v\})) + \ell(B)$, *if $V(B)$ has no simplicial vertex in G.*

Proof. Let G be a block graph and let v the only cut vertex of H in $V(B)$. Notice that H is the graph obtained from G by removing every simplicial vertex belonging to a leaf block of G. By Corollary 1, G has a maximum independent set S such that if B' is any leaf block of G having a simplicial vertex w, then $V(B') \cap S = \{w\}$. Hence if $V(B)$ has a simplicial vertex in G, then $v \notin S$. Therefore, the only vertices of $V(B) \cup L(B)$ in S are those simplicial vertices in a leaf block of G having a vertex in common with $V(B)$ and exactly one of the simplicial vertices of G in $V(B)$, the remaining vertices of S are in $V(G) \setminus (V(B) \cup L(B))$ and the result holds. If $V(B)$ does not have any simplicial vertex of G, then $(V(B) \cup L(B)) \setminus \{v\}$ has in S exactly one vertex for each leaf block of G having a vertex in common with $V(B)$ and v might belong or not to S, thus the second statement holds. □

The following lemma will allow to describe with more precision the structure of those block graphs with prescribed independence number having maximum spectral radius. Recall that two blocks in a graph have at most one vertex in common, which is also a cut vertex.

Lemma 6. *If G is a block graph with maximum spectral radius among all block graphs with independence number α, and B_1 and B_2 are leaf blocks of $G - (\mathcal{L}(G) \cap \mathcal{S}_1(G))$, then $|V(B_1) \cap V(B_2)| = 1$.*

Proof. Suppose, towards a contradiction, that $V(B_1) \cap V(B_2) = \emptyset$. Assume that v_i is the only cut vertex in $V(B_i)$ that does not belong to a leaf block of G, for each $i \in \{1, 2\}$. We are going to split the proof into the three only possible cases. Let x be a principal eigenvector of G having all its coordinates positive.

Case 1: $V(B_i) \cap \mathcal{S}_1(G) \neq \emptyset$ for each $i \in \{1, 2\}$.

Let $S_i = V(B_i) \cap \mathcal{S}_1(G)$ for each $i \in \{1, 2\}$. Consider for instance the graph depicted in Fig. 1 where those blocks playing the roles of B_1 and B_2 are those induced by $\{g, j, k\}$ and $\{i, \ell, m\}$, respectively. In this case $v_1 = k$, $v_2 = \ell$, $S_1 = \{j\}$ and $S_2 = \{m\}$.

By Lemma 5 we know that

$$\alpha = \alpha\left(G - \left(\bigcup_{i=1}^{2}(V(B_i) \cup L(B_i))\right)\right) + \ell_G(B_1) + \ell_G(B_2) + 2.$$

Assume, without losing generality, that $\sum_{a\in S_1} x_a + x_{v_1} \le \sum_{a\in S_2} x_a + x_{v_2}$. We construct a graph G^* from G as follows. We delete every edge sv with $s \in S_1 \cup \{v_1\}$ and $v \in V(B_1) \setminus (S_1 \cup \{v_1\})$ and then we add every edge vw with $v \in V(B_1) \setminus (S_1 \cup \{v_1\})$ and $w \in V(B_2)$. Clearly, G^* is a block graph and its block B' whose vertex set is $(V(B_1) \setminus (S_1 \cup \{v_1\})) \cup V(B_2)$ has at least a simplicial vertex because $V(B_2)$ has a simplicial vertex in G, and the block B'' induced by $S_1 \cup \{v_1\}$ in G^* is a leaf block of G^*. Besides, B' is a leaf block of $G^* - (\mathcal{L}(G^*) \cap \mathcal{S}_1(G^*))$ having a simplicial vertex and $\ell_{G^*}(B') = \ell_G(B_1) + \ell_G(B_2)$. By Lemmas 1 and 3, $\rho(G) < \rho(G^*)$. In virtue of Lemma 5 and Corollary 2 applied to B''

$$\begin{aligned}\alpha(G^*) &= \alpha(G^* - V(B'')) + 1\\ &= \alpha((G^* - V(B'')) \setminus (V(B') \cup L(B'))) + \ell_{G^*}(B') + 2\\ &= \alpha\left(G - \left(\bigcup_{i=1}^{2}(V(B_i) \cup L(B_i))\right)\right) + \ell_G(B_1) + \ell_G(B_2) + 2.\end{aligned}$$

We reach a contradiction.

Case 2: Exactly one of $V(B_1)$ or $V(B_2)$ has a simplicial vertex of G.

Assume, without losing generality, that $V(B_1)$ has at least one simplicial vertex of G and $V(B_2) \cap \mathcal{S}_1(G) = \emptyset$. Let $S = V(B_1) \cap \mathcal{S}_1(G)$

Consider for instance the graph depicted in Fig. 2 where those blocks playing the roles of B_1 and B_2 are those induced by $\{i, \ell, m\}$ and $\{g, j, k\}$, respectively. In this case $v_1 = \ell$, $v_2 = k$ and $S = \{m\}$.

By Lemma 5 we know that

$$\alpha = \alpha\left(G - \left(\left(\bigcup_{i=1}^{2}(V(B_i) \cup L(B_i))\right) \setminus \{v_2\}\right)\right) + \ell_G(B_1) + \ell_G(B_2) + 1.$$

If $x_{v_2} \le \sum_{a\in S} x_a + x_{v_1}$, then the block graph G^* obtained by deleting every edge bv_2 with $b \in (V(B_2) \setminus \{v_2\})$ and by adding every edge bv_1 with $b \in (V(B_2) \setminus \{v_2\})$ satisfies, by Lemma 2, that $\rho(G) < \rho(G^*)$. Notice that B_1 is a block of G^* having at least one simplicial vertex such that $\ell_{G^*}(B_1) = \ell_G(B_1)$, and $B_2' = G^*[(B_2 - v_2) \cup \{v_1\}]$ is a block of G^* having no simplicial vertices such that $\ell_{G^*}(B_2') = \ell_G(B_2)$. Besides, both of B_1' and B_2' are leaf blocks of $G^* \setminus (\mathcal{L}(G^*) \cap \mathcal{S}_1(G^*))$, where $B_1' = B_1$, and thus by Lemma 5

$$\begin{aligned}\alpha(G^*) &= \alpha\left(G^* - \left(\bigcup_{i=1}^{2}(V(B_i') \cup L(B_i'))\right)\right) + \ell_{G^*}(B_1') + \ell_{G^*}(B_2') + 1\\ &= \alpha\left(G - \left(\left(\bigcup_{i=1}^{2}(V(B_i) \cup L(B_i))\right) \setminus \{v_2\}\right)\right) + \ell_G(B_1) + \ell_G(B_2) + 1.\end{aligned}$$

Thus we reach a contradiction.

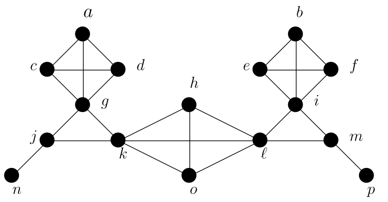

Fig. 3. Block graphs with four leaf blocks, one nonleaf simplicial blocks and two nonsimplicial blocks.

Suppose now that $x_{v_2} \geq \sum_{a \in S} x_a + x_{v_1}$. We construct a block graph G^* from G as follows. We delete every edge sv with $s \in S \cup \{v_1\}$ and $v \in V(B_1) \setminus (S \cup \{v_1\})$, and we add every edge vw with $v \in V(B_1) \setminus (S \cup \{v_1\})$ and $w \in V(B_2)$. Clearly, the block B' of G^* whose vertex set is $V(B_1 - (S \cup \{v_1\})) \cup V(B_2)$ has no simplicial vertex of G^* and $\ell_{G^*}(B') = \ell_G(B_1) + \ell_G(B_2)$, and the block B'' induced in G^* by $S \cup \{v_1\}$ is a leaf block. By Lemmas 1 and 3, $\rho(G) < \rho(G^*)$. By Lemma 5 and Corollary 2

$$\begin{aligned}\alpha(G^*) &= \alpha((G^* - V(B'')) \setminus ((V(B') \cup L(B')) \setminus \{v_2\})) + \ell_{G^*}(B') + 1 \\ &= \alpha\left(G - \left(\left(\bigcup_{i=1}^{2}(V(B_i) \cup L(B_i))\right) \setminus \{v_2\}\right)\right) + \ell_G(B_1) + \ell_G(B_2) + 1.\end{aligned}$$

We reach a contradiction.

Case 3: $V(B_i)$ has no simplicial vertex of G for each $i \in \{1, 2\}$.

Consider for instance the graph depicted in Fig. 3 where those blocks playing the roles of B_1 and B_2 are those induced by $\{g, j, k\}$ and $\{i, \ell, m\}$, respectively. In this case $v_1 = k$ and $v_2 = \ell$.

By Lemma 5 we know that

$$\alpha = \alpha(G) = \alpha\left(G - \left(\left(\bigcup_{i=1}^{2}(V(B_i) \cup L(B_i))\right) \setminus \{v_1, v_2\}\right)\right) + \ell_G(B_1) + \ell_G(B_2).$$

Assume, without losing generality, that $x_{v_1} \geq x_{v_2}$. We transform G into the block graph G^* by deleting every edge v_2u with $u \in V(B_2) \setminus \{v_2\}$ and adding every edge v_1u with $u \in V(B_2) \setminus \{v_2\}$. By Lemma 2, $\rho(G) < \rho(G^*)$. Let define the blocks B'_1 and B'_2 of G^* as those induced by $V(B_1)$ and $V((B_2 - v_2) \cup \{v_1\})$, respectively. In addition, B'_1 and B'_2 are blocks of $G^* - (\mathcal{L}(G^*) \cap \mathcal{S}_1(G^*))$ such that $\ell_{G^*}(B'_i) = \ell_G(B_i)$ for each $i \in \{1, 2\}$. Besides, by Lemma 5,

$$\begin{aligned}\alpha(G^*) &= \alpha\left(G^* - \left(\left(\bigcup_{i=1}^{2}(V(B'_i) \cup L(B'_i))\right) \setminus \{v_1\}\right)\right) + \ell_{G^*}(B'_1) + \ell_{G^*}(B'_2) \\ &= \alpha\left(G - \left(\left(\bigcup_{i=1}^{2}(V(B_i) \cup L(B_i))\right) \setminus \{v_1, v_2\}\right)\right) + \ell_G(B_1) + \ell_G(B_2).\end{aligned}$$

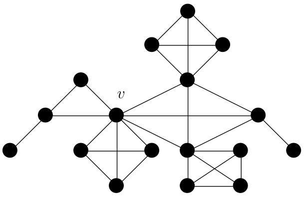

Fig. 4. Block graph G with $G-(\mathcal{L}(G)\cap\mathcal{S}_1(G))$ having all its block sharing the cut vertex v.

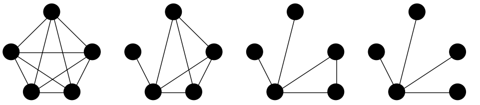

Fig. 5. From left to right we have $P^{\alpha-1}_{5-\alpha+1}$ for every $1\leq\alpha\leq 4$.

Since we reach a contradiction in all of the cases we conclude that every pair of leaf block of $G-(\mathcal{L}(G)\cap\mathcal{S}_1(G))$ have a common cut vertex (see for instance the graph depicted in Fig. 4) and thus every leaf block of $G-(\mathcal{L}(G)\cap\mathcal{S}_1(G))$ have the same common cut vertex in $G-(\mathcal{L}(G)\cap\mathcal{S}_1(G))$. □

The *pineapple* P^p_q is the graph whose vertex set can be partitioned into a clique Q on q vertices and a stable set I on p vertices such that every vertex of I is adjacent to the same vertex in Q (see Fig. 5).

Theorem 2. *Let $G\in\mathcal{G}(n,\alpha)$. Then, $\rho(G)\leq\rho(P^{\alpha-1}_{n-\alpha+1})$. In addition, the equality holds if and only if $G=P^{\alpha-1}_{n-\alpha+1}$.*

Proof. Let G be a block graph with maximum independence set α. Assume that $\alpha\geq 2$, otherwise $G=P^0_n=K_n$. By Lemma 6 either every block of $G-(\mathcal{L}(G)\cap\mathcal{S}_1(G))$ has common cut vertex v (see Fig. 4 for an example), or $G-(\mathcal{L}(G)\cap\mathcal{S}_1(G))=K_r$. Let b be the number of nonleaf blocks having at least one simplicial vertex, let t be the number of leaf blocks sharing the cut vertex v, and let ℓ be the number of leaf blocks of G such that v is not in their vertex sets. By Lemma 5 and Corollary 2, $\alpha=\ell+1$ whenever $b=0$, $t=0$ and $G-(\mathcal{L}(G)\cap\mathcal{S}_1(G))\neq K_r$, or $\alpha=b+t+\ell$, otherwise.

In the sequel, we transform G into G^*, whose vertex sets agree, where v either is the only cut vertex of G^* in every nonleaf block of G^* or is the only simplicial vertex of the only nonleaf block of G^*, we will use b^* to denote the number of nonleaf blocks in G^* having at least one simplicial vertex, and t^* to denote the number of leaf blocks sharing the cut vertex v, and ℓ^* to denote the number of leaf blocks such that v does not belong to them. We will split the proof into four claims.

Claim 1: There exists at most one nonleaf block in G without simplicial vertices.

Suppose, towards a contradiction, that there exist two nonleaf blocks B_1 and B_2 without simplicial vertices. Consider the graph G^* obtained from G by adding every edge v_1v_2 with $v_i \in V(B_i) \setminus \{v\}$ for each $i \in \{1, 2\}$. Clearly, G^* is a block graph. On the one hand, if $b = 0$ and $t = 0$ then $b^* = 1$, whenever there is exactly two nonleaf blocks sharing the cut vertex v, or else $b^* = 0$, $t^* = t$ and $\ell^* = \ell$. On the other hand, $b^* = b$, $t^* = t$ and $\ell^* = \ell$. Hence $\alpha(G^*) = \alpha$. Besides, by Lemma 1 we have $\rho(G) < \rho(G^*)$. We reach a contradiction. Therefore, G has at most one nonleaf block without simplicial vertices.

From Claim 1 we can conclude $\alpha = b + t + \ell$.

Claim 2: There exists at most one nonleaf block B_1

Suppose, towards a contradiction, that there exist two nonleaf blocks B_1 and B_2 in G. By Claim 1 at most one of B_1 and B_2 have no simplicial vertex. First, assume, without lose of generality, that $V(B_2)$ contains no simplicial vertex. Set $S_1 = V(B_1) \cap \mathcal{S}_1(G)$. We transform the graph G into G^* by adding every edge v_1v_2 with $v_i \in V(B_1) \setminus \{v\}$ for every $i \in \{1, 2\}$. Clearly, $b^* = b$, $\ell^* = \ell$ and $t^* = t$. Hence $\alpha(G^*) = \alpha$. In addition, by Lemma 1, $\rho(G) < \rho(G^*)$, reaching a contradiction. Finally, assume that $V(B_i) \cap \mathcal{S}_1(G) \neq \emptyset$ and let $S_i = V(B_i) \cap \mathcal{S}_1(G)$, for each $i \in \{1, 2\}$. Suppose, without losing generality, that $\sum_{u\in S_1} x_u \geq \sum_{u\in S_2} x_u$. We construct the graph G^* from G by deleting every edge xy with $x \in S_2$ and $y \in V(B_2) \setminus (S_2 \cup \{v\})$ and adding every edge yz with $y \in V(B_2) \setminus (S_2 \cup \{v\})$ and $z \in V(B_1) \setminus \{v\}$. Clearly, $b^* = b - 1$, $t^* = t + 1$ and $\ell^* = \ell$. Hence, $\alpha(G^*) = \alpha$. Besides, by Lemma 3, $\rho(G) < \rho(G^*)$, reaching a contradiction.

Claim 3: Every block in G is a leaf block.

Suppose, towards a contradiction, that G has at least a nonleaf block. First assume that B (by Claim 1) is the only nonleaf block in G having no simplicial vertex. Hence, by Claim 2, the remaining blocks are leaf blocks. Let B' be one of those leaf blocks having v' as the only cut vertex of B' in G. By Lemma 1, the graph G^* obtained from G by adding every edge ww' with $w \in V(B) \setminus \{v'\}$ and $w' \in V(B') \setminus \{v'\}$ satisfies $\rho(G) < \rho(G^*)$. In addition, $b^* = 1$, $t^* = t$ and $\ell^* = \ell - 1$. Hence, $\alpha(G^*) = \alpha$, reaching a contradiction.

Assume now that every nonleaf block of G has at least one simplicial vertex. By Claim 2 we conclude that G has only one nonleaf block having at least one simplicial vertex. Hence there exists a leaf block B having u as the only cut vertex of G. Suppose that B' is another leaf block having u' as the only cut vertex of G with $u' \neq u$. Assume first that $x_u \geq x_{u'}$. By Lemma 2, the graph G^* obtained from G by deleting every edge $w'u'$ with $w' \in V(B') \setminus \{u'\}$ and adding every edge $w'u$ with $w' \in V(B') \setminus \{u'\}$, satisfies $\rho(G) < \rho(G^*)$. Notice that, $b^* = b - 1 = 0$, whenever $\ell = 2$, and $b^* = b$ if $\ell > 2$. In both cases $\alpha(G^*) = \alpha$. By symmetry, if $x_u \leq x_{u'}$ applying the analogous transformation we obtain a graph G^* with $\alpha(G^*) = \alpha$ such that $\rho(G) < \rho(G^*)$. We reach a contradiction.

Claim 4: $G = P^{\alpha-1}_{n-\alpha+1}$.

Claim 3 implies that every block in G is a leaf block, sharing a cut vertex u. Hence it remains to prove that at most one block B has at least three vertices. Notice that if every leaf block in G has exactly two vertices, then $G = K_{1,n-1}$. Suppose, towards a contradiction, that B_1 and B_2 are two leaf blocks having at least three vertices. Let $u_i \in V(B_i)$ such that $u_i \neq u$ and let $S_i = V(B_i) \setminus \{u, u_i\}$, for each $i \in \{1, 2\}$. Assume, without losing generality, that $\sum_{s\in S_2} x_s \leq \sum_{s\in S_1} x_s$. Hence, by Lemmas 1 and 3, the graph G^* obtained from G by deleting every edge $u_2 s$ with $s \in S_2$ and adding every edge $u_2 w$ with $w \in S_1 \cup \{v_1\}$, satisfies $\rho(G) < \rho(G^*)$. Besides, clearly $\alpha(G^*) = \alpha$, reaching a contradiction. □

The following lemma gives an upper bound of the spectral radius of the pineapple graph.

Lemma 7. *Let $P^{\alpha-1}_{n-\alpha+1}$ be the pineapple graph with $2 \leq \alpha \leq n-2$. Then*

$$\rho(P^{\alpha-1}_{n-\alpha+1}) \leq \beta - 1 + \frac{\sqrt{(\beta^2-n)^2 + 4(n-\beta)(2\beta-1)} - (\beta^2 - n)}{4\beta - 2}, \tag{1}$$

for $2 \leq \alpha \leq n - \sqrt{n-1}$, and

$$\rho(P^{\alpha-1}_{n-\alpha+1}) \leq \frac{2\sqrt{n-1} + \sqrt{(\alpha-1)\gamma^2 + (n-\alpha)(2-\gamma)}}{2+\gamma}, \tag{2}$$

for $n - \sqrt{n-1} < \alpha \leq n-2$, where $\beta = n - \alpha + 1$ and $\gamma = 1 - \frac{n-\alpha-1}{\sqrt{n-1}}$.

Proof. In [10, Proposition 1.1], it is proved that the characteristic polynomial $p(x) = \det(xI - A)$, where A is the adjacency matrix of $P^{\alpha-1}_{n-\alpha+1}$, is given by

$$p(x) = x^{\alpha-2}(x+1)^{n-\alpha-1}(x^3 - (n-\alpha-1)x^2 - (n-1)x + (\alpha-1)(n-\alpha-1)).$$

Perron-Frobenius implies that $\rho(P^{\alpha-1}_{n-\alpha+1})$ coincides with the maximum positive root of

$$q(x) = x^3 - (n-\alpha-1)x^2 - (n-1)x + (\alpha-1)(n-\alpha-1). \tag{3}$$

We will find an upper bound to the maximum positive root of q. Notice that the pineapple $P^{\alpha-1}_{n-\alpha+1}$ contains $K_{n-\alpha+1}$ and $K_{1,n-1}$ as a subgraph, based on this fact $\max\{n-\alpha, \sqrt{n-1}\} \leq \rho(P^{\alpha-1}_{n-\alpha+1})$ (see [11, Corollary 7] for more details). We will split the task into two cases.

Case 1: $2 \leq \alpha \leq n - \sqrt{n-1}$.

It is easy to see that

$$\rho(P_{n-\alpha+1}^{\alpha-1}) = \beta - 1 + t,$$

where $\beta = n - \alpha + 1$ and t is the maximum positive solution of

$$x^3 + (2\beta - 1)x^2 + (\beta^2 - n)x - (\alpha - 1) = 0.$$

Since $t > 0$, we have that

$$(2\beta - 1)t^2 + (\beta^2 - n)t - (\alpha - 1) < 0.$$

It follows immediately that

$$t \leq \frac{\sqrt{(\beta^2 - n)^2 + 4(n - \beta)(2\beta - 1)} - (\beta^2 - n)}{4\beta - 2}.$$

Finally, we conclude

$$\rho(P_{n-\alpha+1}^{\alpha-1}) \leq \beta - 1 + \frac{\sqrt{(\beta^2 - n)^2 + 4(n - \beta)(2\beta - 1)} - (\beta^2 - n)}{4\beta - 2}.$$

Case 2: $n - \sqrt{n-1} < \alpha \leq n - 2$.

It is easy to see that

$$\rho(P_{n-\alpha+1}^{\alpha-1}) = \sqrt{n-1} + t,$$

where t is the maximum positive solution of

$$x^3 + \sqrt{n-1}(2 + \gamma)\, x^2 + 2(n - 1)\gamma\, x - (n - \alpha)(n - \alpha - 1) = 0,$$

where $\gamma = 1 - \frac{n-\alpha-1}{\sqrt{n-1}}$. Since $t > 0$, we see that

$$\sqrt{n-1}(2 + \gamma)\, t^2 + 2(n - 1)\gamma\, t - (n - \alpha)(n - \alpha - 1) < 0.$$

It follows immediately that

$$t \leq \frac{\sqrt{(\alpha - 1)\gamma^2 + (n - \alpha)(2 - \gamma)} - \sqrt{n-1}\gamma}{2 + \gamma}.$$

Finally, we conclude

$$\rho(P_{n-\alpha+1}^{\alpha-1}) \le \sqrt{n-1} + \frac{\sqrt{(\alpha-1)\gamma^2 + (n-\alpha)(2-\gamma)} - \sqrt{n-1}\gamma}{2+\gamma} \tag{4}$$

$$= \frac{2\sqrt{n-1} + \sqrt{(\alpha-1)\gamma^2 + (n-\alpha)(2-\gamma)}}{2+\gamma}. \quad \square$$

Remark 1. In this remark, we compare the bounds for the spectral radius $\rho(P_{n-\alpha+1}^{\alpha-1})$ obtained in Lemma 7 with those found in [11, Corollaries 7 and 8].

Under the assumption $2 \le \alpha \le n - \sqrt{n-1}$, we have

$$\rho(P_{n-\alpha+1}^{\alpha-1}) \le \beta - 1 + \frac{\sqrt{(\beta^2-n)^2 + 4(n-\beta)(2\beta-1)} - (\beta^2-n)}{4\beta-2}.$$

By the Mean Value Theorem, we see that

$$\frac{\sqrt{(\beta^2-n)^2 + 4(n-\beta)(2\beta-1)} - (\beta^2-n)}{4\beta-2} = \frac{4(n-\beta)(2\beta-1)}{(4\beta-2)2\sqrt{\xi}} = \frac{(n-\beta)}{\sqrt{\xi}},$$

where $(\beta^2-n)^2 < \xi < (\beta^2-n)^2 + 4(n-\beta)(2\beta-1)$. It follows that

$$\beta - 1 + \frac{\sqrt{(\beta^2-n)^2 + 4(n-\beta)(2\beta-1)} - (\beta^2-n)}{4\beta-2} < \beta - 1 + \frac{(n-\beta)}{\beta^2-n}.$$

Hence the bound (1) refines the one present in [11, Corollaries 7].

We now turn to the case $n - \sqrt{n-1} < \alpha \le n-2$. By (4), we have

$$\rho(P_{n-\alpha+1}^{\alpha-1}) \le \sqrt{n-1} + \frac{\sqrt{(\alpha-1)\gamma^2 + (n-\alpha)(2-\gamma)} - \sqrt{n-1}\gamma}{2+\gamma}$$

By the Mean Value Theorem, we see that

$$\frac{\sqrt{(\alpha-1)\gamma^2 + (n-\alpha)(2-\gamma)} - \sqrt{n-1}\gamma}{2+\gamma} = \frac{(n-\alpha)(1-\gamma)}{2\sqrt{\xi}},$$

where $(n-1)\gamma^2 < \xi < (\alpha-1)\gamma^2 + (n-\alpha)(2-\gamma)$. It follows that

$$\sqrt{n-1} + \frac{\sqrt{(\alpha-1)\gamma^2 + (n-\alpha)(2-\gamma)} - \sqrt{n-1}\gamma}{2+\gamma} < \sqrt{n-1} + \frac{(n-\alpha)(1-\gamma)}{2\sqrt{n-1}\gamma}.$$

Thus the bound (2) refines the one presented in [11, Corollaries 8].

Corollary 3. *Let $G \in \mathcal{G}(n,\alpha)$. Then,*

$$\rho(G) \le \beta - 1 + \frac{\sqrt{(\beta^2-n)^2 + 4(n-\beta)(2\beta-1)} - (\beta^2-n)}{4\beta-2}, \tag{5}$$

for $2 \leq \alpha \leq n - \sqrt{n-1}$, *and*

$$\rho(G) \leq \frac{2\sqrt{n-1} + \sqrt{(\alpha-1)\gamma^2 + (n-\alpha)(2-\gamma)}}{2+\gamma}, \tag{6}$$

for $n - \sqrt{n-1} < \alpha \leq n-2$, *where* $\beta = n - \alpha + 1$ *and* $\gamma = 1 - \frac{n-\alpha-1}{\sqrt{n-1}}$.

Declaration of competing interest

There is no competing interest.

Acknowledgements

The authors would like to thank the anonymous referee for generous observations and comments that we have made ample use of in the paper.

References

[1] R.A. Brualdi, E.S. Solheid, On the spectral radius of complementary acyclic matrices of zeros and ones, SIAM J. Algebraic Discrete Methods 7 (1986) 265–272.
[2] D. Stevanoviċ, Spectral Radius of Graphs, Academic Press, 2014.
[3] C. Ji, M. Lu, On the spectral radius of trees with given independence number, Linear Algebra Appl. 488 (2016) 102–108.
[4] H. Lu, Y. Lin, Maximum spectral radius of graphs with given connectivity, minimum degree and independence number, J. Discret. Algorithms 31 (2015) 113–119.
[5] R.B. Bapat, Graphs and Matrices, 2nd edition, Universitext, Springer/Hindustan Book Agency, London/New Delhi, 2014.
[6] L. Lováz, J. Pelikán, On the eigenvalues of trees, Period. Math. Hung. 3 (1973) 175–182.
[7] P. Rowlinson, More on graph perturbations, Bull. Lond. Math. Soc. 22 (1990) 209–216.
[8] B. Wu, E. Xiao, Y. Hong, The spectral radius of trees on k pendant vertices, Linear Algebra Appl. 395 (2005) 343–349.
[9] R.B. Bapat, S. Roy, On the adjacency matrix of a block graph, Linear Multilinear Algebra 62 (2014) 406–418.
[10] H. Topcu, S. Sorgun, W.H. Haemers, On the spectral characterization of pineapple graphs, Linear Algebra Appl. 507 (2016) 267–273.
[11] H. Liu, M. Lu, F. Tian, On the spectral radius of graphs with cut edges, Linear Algebra Appl. 389 (2004) 139–145.