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A note about the norm of the sum and the anticommutator of two orthogonal projections

Cristian Conde

Instituto de Ciencias, Universidad Nacional de Gral. Sarmiento, J.M. Gutierrez 1150, (B1613GSX) Los Polvorines, Argentina

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ABSTRACT

In this note, we prove that for any two orthogonal projections $P_{\mathcal{T}}, P_{\mathcal{S}}$ on a Hilbert the well-known norm formulas

$$\|P_{\mathcal{T}} + P_{\mathcal{S}}\| = 1 + \|P_{\mathcal{T}} P_{\mathcal{S}}\|,$$

unless $P_{\mathcal{T}} = P_{\mathcal{S}} = 0$ and

$$\|P_{\mathcal{T}} P_{\mathcal{S}} + P_{\mathcal{S}} P_{\mathcal{T}}\| = \|P_{\mathcal{T}} P_{\mathcal{S}}\|^2 + \|P_{\mathcal{T}} P_{\mathcal{S}}\|,$$

can be derived from each other. Such result is obtained from the relation between the spectra of the sum and product of any two idempotents in a Banach algebra. Applications of our results are given.

1. Introduction

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be an inner product space over the complex numbers field $\mathbb{C}$. Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators acting on $\mathcal{H}$ with the corresponding norm $\|\cdot\|$. The symbol I stands for the identity operator and for any $T \in \mathcal{B}(\mathcal{H})$ we consider T^* its adjoint. T is called a selfadjoint operator if $T = T^*$.

For each $T \in \mathcal{B}(\mathcal{H})$, we denote its spectrum by $\sigma(T)$, that is, $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$ and the numerical range $W(T)$ is the image of the unit sphere of $\mathcal{H}$ under the quadratic form $x \to \langle Tx, x\rangle$. More precisely,

$$W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}. \tag{1.1}$$

E-mail address: cconde@ungs.edu.ar.

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Thus the numerical range of an operator is a subset of the complex plane whose geometrical properties should say something about that operator. By the Toeplitz-Hausdorff's Theorem $W(T)$ is a convex set.

Then, for any T in $\mathcal{B}(\mathcal{H})$ we define the numerical radius of T,

$$\omega(T) = \sup\{|\lambda| : \lambda \in W(T)\}. \tag{1.2}$$

It is well known that $\omega(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$, and we have for all $T \in \mathcal{B}(\mathcal{H})$,

$$\frac{1}{2}\|T\| \leq \omega(T) \leq \|T\|. \tag{1.3}$$

Thus, the usual operator norm and the numerical radius norm are equivalent. The inequalities in (1.3) are sharp. If $T^2 = 0$, then the first inequality becomes an equality, while the second inequality becomes an equality if T is normal.

A linear operator defined on $\mathcal{H}$, such that $P^2 = P$ is called a projection. Such operators are not necessarily bounded, since on every infinite dimensional Hilbert space there exist unbounded examples of projections, see [4]. Recall that $P \in \mathcal{B}(\mathcal{H})$ is an orthogonal projection on $\mathcal{H}$ if and only if $P^2 = P = P^*$. For a closed subspace $\mathcal{S}$ of $\mathcal{H}$, $P_{\mathcal{S}}$ denotes the orthogonal projection from $\mathcal{H}$ onto $\mathcal{S}$. Orthogonal projections $P_{\mathcal{S}}$ and $P_{\mathcal{T}}$ are called mutually orthogonal if $P_{\mathcal{S}}P_{\mathcal{T}} = 0$. This is equivalent to $\mathcal{S}$ and $\mathcal{T}$ being orthogonal as subspaces of $\mathcal{H}$. The sum is an orthogonal projection only if $P_{\mathcal{S}}$ and $P_{\mathcal{T}}$ are orthogonal to each other. The composite $P_{\mathcal{S}}P_{\mathcal{T}}$ is generally not a projection; in fact, the composite is a projection if and only if the two projections commute.

The next statement collects a classical result about the norm of the difference and sum of two orthogonal projections.

Theorem 1.1. *Let $P_{\mathcal{T}}, P_{\mathcal{S}}$ be orthogonal projections on $\mathcal{H}$. Then the following assertions hold:*

$$\|P_{\mathcal{T}} - P_{\mathcal{S}}\| \leq 1 \leq \|P_{\mathcal{T}} + P_{\mathcal{S}}\| = 1 + \|P_{\mathcal{T}}P_{\mathcal{S}}\|, \tag{1.4}$$

if $P_{\mathcal{T}} \neq 0$ or $P_{\mathcal{S}} \neq 0$.

The last equality is due to Duncan and Taylor ([6]) in the study of norm inequalities for C*-algebras

$$\|P_{\mathcal{S}} + P_{\mathcal{T}}\| = 1 + \|P_{\mathcal{S}}P_{\mathcal{T}}\|. \tag{1.5}$$

An algebraic proof of it is in Vidav's paper [20] and finally in [7], Fujii and Nakamoto gave another proof from estimations of $P_{\mathcal{S}} - P_{\mathcal{T}}$.

Recently, Walters obtained the following anticommutator norm formula for orthogonal projections. Specifically, the norm of the anticommutator $\|P_{\mathcal{T}}P_{\mathcal{S}} + P_{\mathcal{S}}P_{\mathcal{T}}\|$ is a simple quadratic function of the norm $\|P_{\mathcal{T}}P_{\mathcal{S}}\|$.

Theorem 1.2. *[[22], Theorem 1.3] If $P_{\mathcal{T}}, P_{\mathcal{S}}$ are two orthogonal projections on Hilbert space $\mathcal{H}$, then*

$$\|P_{\mathcal{T}}P_{\mathcal{S}} + P_{\mathcal{S}}P_{\mathcal{T}}\| = \|P_{\mathcal{T}}P_{\mathcal{S}}\|^2 + \|P_{\mathcal{T}}P_{\mathcal{S}}\|. \tag{1.6}$$

The remainder of the paper is organized as follows. Section 2 we prove that (1.5) and (1.6) are equivalent and we present new norm inequalities between the sum and the anticommutator of two orthogonal projections. Section, 3 is devoted to obtain new norm inequalities and finally, in Section 4 we conclude this manuscript with bounds for the commutator associated to two orthogonal projections, we give a characterization of a pair $P_{\mathcal{T}}, P_{\mathcal{S}}$ of orthogonal projections with numerical radius of minimum or maximum possible value and we also study the relationship between the numerical radius of a product of two orthogonal projections and the norm of the sum and difference of such projections.

2. Main result

In this section, we present a proof of equality (1.5) using the identity (1.6) and viceversa. To obtain this statement we need the relation between the spectra of the sum and product of any two idempotents in a Banach algebra. Recall that any orthogonal projection is an idempotent element of $\mathcal{B}(\mathcal{H})$.

Theorem 2.1. *[[2], Theorem 1] If $P_\mathcal{T}$ and $P_\mathcal{S}$ are orthogonal projections on $\mathcal{H}$, then*

$$\sigma(P_\mathcal{T} + P_\mathcal{S}) - \{0, 1, 2\} = \{1 \pm \alpha^{1/2} : \alpha \in \sigma(P_\mathcal{T} P_\mathcal{S}) - \{0, 1\}\}. \tag{2.1}$$

Theorem 2.2. *Let $P_\mathcal{T}, P_\mathcal{S}$ be orthogonal projections on $\mathcal{H}$. Then the equalities (1.5) and (1.6) each follows one from the other.*

Proof. If $P_\mathcal{T} = P_\mathcal{S} = 0$ then the equality (1.6) is obvious. So we consider that $P_\mathcal{T} \neq 0$ or $P_\mathcal{S} \neq 0$.

Now the proof shall be divided into two cases. For the first case we use the following inequality,

$$\begin{aligned} \|P_\mathcal{T} P_\mathcal{S}\|^2 + \|P_\mathcal{T} P_\mathcal{S}\| &= \|P_\mathcal{T} + P_\mathcal{S} + P_\mathcal{T} P_\mathcal{S} + P_\mathcal{S} P_\mathcal{T}\| - \|P_\mathcal{T} + P_\mathcal{S}\| \\ &\leq \|P_\mathcal{T} P_\mathcal{S} + P_\mathcal{S} P_\mathcal{T}\|, \end{aligned} \tag{2.2}$$

which is an immediate consequence of (1.5).

First, if $\|P_\mathcal{T} + P_\mathcal{S}\| = 1$ or $\|P_\mathcal{T} + P_\mathcal{S}\| = 2$ then $\|P_\mathcal{T} P_\mathcal{S}\| = 0$ or $\|P_\mathcal{T} P_\mathcal{S}\| = 1$. In both cases by (2.2) and the triangle inequality we conclude that (1.6) holds.

Second, it remains to prove the desired equality when $1 < \|P_\mathcal{T} + P_\mathcal{S}\| < 2$.

$$\|P_\mathcal{T} P_\mathcal{S} + P_\mathcal{S} P_\mathcal{T}\| = \|(P_\mathcal{T} + P_\mathcal{S})^2 - (P_\mathcal{T} + P_\mathcal{S})\| = \|p(P_\mathcal{T} + P_\mathcal{S})\|, \tag{2.3}$$

where $p(x) = x^2 - x$. By functional calculus, as $P_\mathcal{T} + P_\mathcal{S}$ is a bounded self-adjoint linear operator and p is a polynomial then

$$\|p(P_\mathcal{T} + P_\mathcal{S})\| = \sup_{\lambda \in \sigma(P_\mathcal{T} + P_\mathcal{S})} |p(t)|.$$

Since $P_\mathcal{T} + P_\mathcal{S}$ is a positive operator then $\|P_\mathcal{T} + P_\mathcal{S}\| \in \sigma(P_\mathcal{T} + P_\mathcal{S})$ and from the Theorem 2.1 we have that

$$\begin{aligned} \|P_\mathcal{T} P_\mathcal{S} + P_\mathcal{S} P_\mathcal{T}\| &= \sup_{\lambda \in \sigma(P_\mathcal{T} + P_\mathcal{S})} |p(t)| \\ &= p(\|P_\mathcal{T} + P_\mathcal{S}\|) = \|P_\mathcal{T} + P_\mathcal{S}\|^2 - \|P_\mathcal{T} + P_\mathcal{S}\| \\ &= (1 + \|P_\mathcal{T} P_\mathcal{S}\|)^2 - (1 + \|P_\mathcal{T} P_\mathcal{S}\|) \\ &= \|P_\mathcal{T} P_\mathcal{S}\|^2 + \|P_\mathcal{T} P_\mathcal{S}\|. \end{aligned}$$

We suppose that $P_\mathcal{T} \neq 0$ and $P_\mathcal{S} \neq 0$. It is well-known that $1 \leq \|P_\mathcal{T} + P_\mathcal{S}\| \leq 1 + \|P_\mathcal{T} P_\mathcal{S}\|$ (see for example [10]). We assume that $\|P_\mathcal{T} + P_\mathcal{S}\| = 1 + \alpha$ where $\alpha \in [0, \|P_\mathcal{T} P_\mathcal{S}\|]$.

If $\|P_\mathcal{T} + P_\mathcal{S}\| = 1$ by Proposition 2.1 in [5] we conclude that $P_\mathcal{T} + P_\mathcal{S}$ is an orthogonal projection. Hence,

$$P_\mathcal{T} + P_\mathcal{S} = (P_\mathcal{T} + P_\mathcal{S})^2 = P_\mathcal{T} + P_\mathcal{S} + P_\mathcal{T} P_\mathcal{S} + P_\mathcal{S} P_\mathcal{T},$$

and we conclude that $0 = \|P_\mathcal{T} P_\mathcal{S} + P_\mathcal{S} P_\mathcal{T}\| = \|P_\mathcal{T} P_\mathcal{S}\|^2 + \|P_\mathcal{T} P_\mathcal{S}\|$, then $\|P_\mathcal{T} P_\mathcal{S}\| = 0$ and $\|P_\mathcal{T} + P_\mathcal{S}\| = 1 + \|P_\mathcal{T} P_\mathcal{S}\|$.

On the other hand, if $\|P_{\mathcal{T}} + P_{\mathcal{S}}\| = 2$ then $2 = \|P_{\mathcal{T}} + P_{\mathcal{S}}\| \leq \|P_{\mathcal{T}}\| + \|P_{\mathcal{S}}\| = 2$ and hence $\|P_{\mathcal{T}} + P_{\mathcal{S}}\| = \|P_{\mathcal{T}}\| + \|P_{\mathcal{S}}\|$. Then, we have $\|P_{\mathcal{T}}P_{\mathcal{S}}\| = \|P_{\mathcal{T}}\|\|P_{\mathcal{S}}\| = 1$ (see [12]) and

$$\|P_{\mathcal{T}} + P_{\mathcal{S}}\| = 1 + \|P_{\mathcal{T}}P_{\mathcal{S}}\|.$$

Finally, we consider that $1 < \|P_{\mathcal{T}} + P_{\mathcal{S}}\| < 2$.

$$\begin{aligned}\|P_{\mathcal{T}}P_{\mathcal{S}}\|^2 + \|P_{\mathcal{T}}P_{\mathcal{S}}\| = \|P_{\mathcal{T}}P_{\mathcal{S}} + P_{\mathcal{S}}P_{\mathcal{T}}\| &= \sup_{\lambda\in\sigma(P_{\mathcal{T}}+P_{\mathcal{S}})} |p(t)| \\ &= p(\|P_{\mathcal{T}} + P_{\mathcal{S}}\|) = \|P_{\mathcal{T}} + P_{\mathcal{S}}\|^2 - \|P_{\mathcal{T}} + P_{\mathcal{S}}\| \\ &= (1+\alpha)^2 - (1+\alpha) = \alpha^2 + \alpha \\ &\leq \|P_{\mathcal{T}}P_{\mathcal{S}}\|^2 + \|P_{\mathcal{T}}P_{\mathcal{S}}\|.\end{aligned}$$

As $\alpha^2 + \alpha = \|P_{\mathcal{T}}P_{\mathcal{S}}\|^2 + \|P_{\mathcal{T}}P_{\mathcal{S}}\|$ and $\alpha \in (0, \|P_{\mathcal{T}}P_{\mathcal{S}}\|]$ we conclude that $\alpha = \|P_{\mathcal{T}}P_{\mathcal{S}}\|$.

This finishes the proof. □

3. Applications

As applications of our previous results, we have the following corollaries. Now we deduce that the sum and the anticommutator of two orthogonal projections satisfy the triangle equality. Kittaneh provided necessary and sufficient conditions on a finite sequence of positive operators in order to obtain the triangle equality. Although $P_{\mathcal{T}}P_{\mathcal{S}} + P_{\mathcal{S}}P_{\mathcal{T}}$ is not a positive operator, but self-adjoint, we have a result similar to Kittaneh for the anticommutator and the sum of two orthogonal projections.

Proposition 3.1. *Let $P_{\mathcal{T}}, P_{\mathcal{S}}$ be orthogonal projections on $\mathcal{H}$. Then*
(1)

$$\|P_{\mathcal{T}}P_{\mathcal{S}} + P_{\mathcal{S}}P_{\mathcal{T}} + P_{\mathcal{T}} + P_{\mathcal{S}}\| = \|P_{\mathcal{T}}P_{\mathcal{S}} + P_{\mathcal{S}}P_{\mathcal{T}}\| + \|P_{\mathcal{T}} + P_{\mathcal{S}}\|. \tag{3.1}$$

(2)

$$\|(P_{\mathcal{T}}P_{\mathcal{S}} + P_{\mathcal{S}}P_{\mathcal{T}})(P_{\mathcal{T}} + P_{\mathcal{S}})\| = \|P_{\mathcal{T}}P_{\mathcal{S}} + P_{\mathcal{S}}P_{\mathcal{T}}\|\|P_{\mathcal{T}} + P_{\mathcal{S}}\|. \tag{3.2}$$

Proof. (1) It is an immediate consequence of (1.5) and (1.6), since

$$\begin{aligned}\|P_{\mathcal{T}}P_{\mathcal{S}} + P_{\mathcal{S}}P_{\mathcal{T}}\| + \|P_{\mathcal{T}} + P_{\mathcal{S}}\| &= 1 + 2\|P_{\mathcal{S}}P_{\mathcal{T}}\| + \|\|P_{\mathcal{S}}P_{\mathcal{T}}\|^2 \\ &= (1 + \|P_{\mathcal{S}}P_{\mathcal{T}}\|)^2 \\ &= \|P_{\mathcal{T}} + P_{\mathcal{S}}\|^2 \\ &= \|P_{\mathcal{T}}P_{\mathcal{S}} + P_{\mathcal{S}}P_{\mathcal{T}} + P_{\mathcal{T}} + P_{\mathcal{S}}\|.\end{aligned}$$

(2) It is an immediate consequence of item (1) and Corollary 2.2 in [1]. □

In [11] (or [8]) the well-known Krein–Krasnoselskii–Milman equality (KKME) was obtained. Let $P_{\mathcal{T}}, P_{\mathcal{S}}$ be orthogonal projections on $\mathcal{H}$, then the following assertion holds:

$$\|P_{\mathcal{S}} - P_{\mathcal{T}}\| = \max\{\|P_{\mathcal{S}}(I - P_{\mathcal{T}})\|, \|P_{\mathcal{T}}(I - P_{\mathcal{S}})\|\}. \tag{3.3}$$

Now, we give an alternative proof of the KKME as an immediate consequence of Duncan-Taylor equality.

Proposition 3.2. *Let $P_{\mathcal{T}}, P_{\mathcal{S}}$ be orthogonal projections on $\mathcal{H}$, then*

$$\|P_{\mathcal{S}} - P_{\mathcal{T}}\| = \max\{\|P_{\mathcal{S}}(I - P_{\mathcal{T}})\|, \|P_{\mathcal{T}}(I - P_{\mathcal{S}})\|\}, \tag{3.4}$$

Proof. By (1.5) we have that

$$\begin{aligned}\|P_{\mathcal{S}}(I - P_{\mathcal{T}})\| &= \|P_{\mathcal{S}} + (I - P_{\mathcal{T}})\| - 1 \leq \|P_{\mathcal{S}} + (I - P_{\mathcal{T}}) - I\| \\ &= \|P_{\mathcal{S}} - P_{\mathcal{T}}\|.\end{aligned} \tag{3.5}$$

Similarly, we obtain

$$\|P_{\mathcal{T}}(I - P_{\mathcal{S}})\| \leq \|P_{\mathcal{T}} - P_{\mathcal{S}}\|. \tag{3.6}$$

Applying (3.5) and (3.6) yields

$$\max\{\|P_{\mathcal{S}}(I - P_{\mathcal{T}})\|, \|P_{\mathcal{T}}(I - P_{\mathcal{S}})\|\} \leq \|P_{\mathcal{S}} - P_{\mathcal{T}}\|. \tag{3.7}$$

It remains to show that this lower bound is sharp. We note that $P_{\mathcal{S}} - P_{\mathcal{T}} = P_{\mathcal{S}}(I - P_{\mathcal{T}}) - (I - P_{\mathcal{S}})P_{\mathcal{T}}$ and $P_{\mathcal{S}}(I - P_{\mathcal{T}}), (I - P_{\mathcal{S}})P_{\mathcal{T}}$ have orthogonal ranges. Then

$$\begin{aligned}\|P_{\mathcal{S}} - P_{\mathcal{T}}\|^2 &= \|P_{\mathcal{S}}(I - P_{\mathcal{T}}) - (I - P_{\mathcal{S}})P_{\mathcal{T}}\|^2 \\ &= \|P_{\mathcal{S}}(I - P_{\mathcal{T}}) + (I - P_{\mathcal{S}})P_{\mathcal{T}}\|^2 \\ &\leq \max\{\|P_{\mathcal{S}}(I - P_{\mathcal{T}})\|^2, \|P_{\mathcal{T}}(I - P_{\mathcal{S}})\|^2\},\end{aligned} \tag{3.8}$$

in last inequality we use Remark 2.4 in [3].

This completes the proof. □

Now, we start by presenting inequalities between the norm of $P_{\mathcal{S}} + P_{\mathcal{T}}$ and $P_{\mathcal{S}} - P_{\mathcal{T}}$.

Proposition 3.3. *Let $P_{\mathcal{T}}, P_{\mathcal{S}}$ be orthogonal projections on $\mathcal{H}$, then*

$$\|P_{\mathcal{T}}P_{\mathcal{S}} + P_{\mathcal{S}}P_{\mathcal{T}}\| \geq \frac{\|P_{\mathcal{S}} + P_{\mathcal{T}}\|^2 - \|P_{\mathcal{T}} - P_{\mathcal{S}}\|^2}{2}. \tag{3.9}$$

Proof. To prove (3.9), observe that

$$\begin{aligned}2\|P_{\mathcal{T}}P_{\mathcal{S}} + P_{\mathcal{S}}P_{\mathcal{T}}\| &= \|(P_{\mathcal{S}} + P_{\mathcal{T}})^2 - (P_{\mathcal{S}} - P_{\mathcal{T}})^2\| \\ &\geq \big|\|(P_{\mathcal{S}} + P_{\mathcal{T}})^2\| - \|(P_{\mathcal{S}} - P_{\mathcal{T}})^2\|\big| \\ &= \|P_{\mathcal{S}} + P_{\mathcal{T}}\|^2 - \|P_{\mathcal{S}} - P_{\mathcal{T}}\|^2. \quad \square\end{aligned}$$

We now derive operator norm inequalities comparing the difference and the anticommutator of two orthogonal projections.

Corollary 3.1. *Let $P_{\mathcal{T}}, P_{\mathcal{S}}$ be orthogonal projections on $\mathcal{H}$, then*

$$\begin{aligned}\|P_{\mathcal{T}} + P_{\mathcal{S}}\| - \|P_{\mathcal{T}} - P_{\mathcal{S}}\| &\leq \|P_{\mathcal{T}} + P_{\mathcal{S}}\| - \|P_{\mathcal{T}} + P_{\mathcal{S}} - 2P_{\mathcal{T}}P_{\mathcal{S}}\| \\ &\leq \|P_{\mathcal{T}} + P_{\mathcal{S}}\| - \|P_{\mathcal{T}} - P_{\mathcal{S}}\|^2 \\ &\leq \frac{\|P_{\mathcal{S}} + P_{\mathcal{T}}\|^2 - \|P_{\mathcal{T}} - P_{\mathcal{S}}\|^2}{2} \\ &\leq \|P_{\mathcal{T}}P_{\mathcal{S}} + P_{\mathcal{S}}P_{\mathcal{T}}\|.\end{aligned} \tag{3.10}$$

Proof. To prove these inequalities, observe that

$$\begin{aligned}\|P_{\mathcal{T}}-P_{\mathcal{S}}\|^2 &= \min\{\|P_{\mathcal{T}}-P_{\mathcal{S}}\|^2, \|P_{\mathcal{T}}+P_{\mathcal{S}}\|^2\} \\ &= \frac{1}{2}(\|P_{\mathcal{T}}+P_{\mathcal{S}}\|^2+\|P_{\mathcal{T}}-P_{\mathcal{S}}\|^2-\left|\|P_{\mathcal{T}}+P_{\mathcal{S}}\|^2-\|P_{\mathcal{T}}-P_{\mathcal{S}}\|^2\right|) \\ &= \frac{1}{2}(\|(P_{\mathcal{T}}+P_{\mathcal{S}})^2\|+\|(P_{\mathcal{T}}-P_{\mathcal{S}})^2\|-\left|\|P_{\mathcal{T}}+P_{\mathcal{S}}\|^2-\|P_{\mathcal{T}}-P_{\mathcal{S}}\|^2\right|) \\ &\geq \frac{1}{2}(\|(P_{\mathcal{T}}+P_{\mathcal{S}})^2+(P_{\mathcal{T}}-P_{\mathcal{S}})^2\|-\left|\|P_{\mathcal{T}}+P_{\mathcal{S}}\|^2-\|P_{\mathcal{T}}-P_{\mathcal{S}}\|^2\right|) \\ &= \|P_{\mathcal{T}}+P_{\mathcal{S}}\|-\frac{\left|\|P_{\mathcal{T}}+P_{\mathcal{S}}\|^2-\|P_{\mathcal{T}}-P_{\mathcal{S}}\|^2\right|}{2}.\end{aligned}$$

Consequently,

$$\frac{\|P_{\mathcal{T}}+P_{\mathcal{S}}\|^2-\|P_{\mathcal{T}}-P_{\mathcal{S}}\|^2}{2} \geq \|P_{\mathcal{T}}+P_{\mathcal{S}}\|-\|P_{\mathcal{T}}-P_{\mathcal{S}}\|^2. \tag{3.11}$$

The other inequalities follow from (3.9) and Corollary 2 in [16]. □

In [17], Klaja linked the numerical radius and the spectral radius of $P_{\mathcal{T}}P_{\mathcal{S}}$ by the following formula:

$$\omega(P_{\mathcal{S}}P_{\mathcal{T}}) = \frac{1}{2}\left(\sqrt{r(P_{\mathcal{S}}P_{\mathcal{T}})}+r(P_{\mathcal{S}}P_{\mathcal{T}})\right).$$

Recall that the *spectral radius* $r(T)$ of $T \in \mathcal{B}(\mathcal{H})$ is defined as $r(T) = \sup\{|z|, z \in \sigma(T)\}$. On the other hand, as $P_{\mathcal{T}}P_{\mathcal{S}}+P_{\mathcal{S}}P_{\mathcal{T}}$ is a selfadjoint operator and $\omega(P_{\mathcal{T}}P_{\mathcal{S}}) = \omega(P_{\mathcal{S}}P_{\mathcal{T}})$, we have

$$\|P_{\mathcal{T}}P_{\mathcal{S}}+P_{\mathcal{S}}P_{\mathcal{T}}\| = \omega(P_{\mathcal{T}}P_{\mathcal{S}}+P_{\mathcal{S}}P_{\mathcal{T}}) \leq \omega(P_{\mathcal{T}}P_{\mathcal{S}})+\omega(P_{\mathcal{S}}P_{\mathcal{T}}) = 2\omega(P_{\mathcal{T}}P_{\mathcal{S}}).$$

Now we prove that the above inequality is actually an equality and we obtain a new expression for the norm of the anticommutator associated two orthogonal projections.

Theorem 3.1. *Let $P_{\mathcal{T}}, P_{\mathcal{S}}$ be orthogonal projections on $\mathcal{H}$, then*

$$\|P_{\mathcal{T}}P_{\mathcal{S}}+P_{\mathcal{S}}P_{\mathcal{T}}\| = \sqrt{r(P_{\mathcal{S}}P_{\mathcal{T}})}+r(P_{\mathcal{S}}P_{\mathcal{T}}). \tag{3.12}$$

Proof. We start by remarking that

$$\sigma(P_{\mathcal{S}}P_{\mathcal{T}}P_{\mathcal{S}})-\{0\} = \sigma(P_{\mathcal{T}}P_{\mathcal{S}}P_{\mathcal{S}})-\{0\} = \sigma(P_{\mathcal{T}}P_{\mathcal{S}})-\{0\},$$

and $P_{\mathcal{S}}P_{\mathcal{T}}P_{\mathcal{S}} = (P_{\mathcal{T}}P_{\mathcal{S}})^*(P_{\mathcal{T}}P_{\mathcal{S}})$ is a selfadjoint operator. Then,

$$r(P_{\mathcal{S}}P_{\mathcal{T}}) = r(P_{\mathcal{S}}P_{\mathcal{T}}P_{\mathcal{S}}) = \|P_{\mathcal{S}}P_{\mathcal{T}}P_{\mathcal{S}}\| = \|(P_{\mathcal{T}}P_{\mathcal{S}})^*(P_{\mathcal{T}}P_{\mathcal{S}})\| = \|P_{\mathcal{T}}P_{\mathcal{S}}\|^2.$$

Replacing in (1.6) completes the proof. □

Remark 3.1. In view of (1.6), it is evident that inequality (3.10) is a refinement of (1.3) since

$$\begin{aligned}\|P_{\mathcal{T}}P_{\mathcal{S}}\| &\leq \|P_{\mathcal{T}}+P_{\mathcal{S}}\|-\|P_{\mathcal{T}}-P_{\mathcal{S}}\| \leq \|P_{\mathcal{T}}+P_{\mathcal{S}}\|-\|P_{\mathcal{T}}+P_{\mathcal{S}}-2P_{\mathcal{T}}P_{\mathcal{S}}\| \\ &\leq \|P_{\mathcal{T}}+P_{\mathcal{S}}\|-\|P_{\mathcal{T}}-P_{\mathcal{S}}\|^2 \leq \frac{\|P_{\mathcal{S}}+P_{\mathcal{T}}\|^2-\|P_{\mathcal{T}}-P_{\mathcal{S}}\|^2}{2} \\ &\leq \|P_{\mathcal{T}}P_{\mathcal{S}}+P_{\mathcal{S}}P_{\mathcal{T}}\| = 2\omega(P_{\mathcal{T}}P_{\mathcal{S}}).\end{aligned} \tag{3.13}$$

On the other hand, from (1.3), Theorem 1 in [13] and (1.5) we have that

$$2\omega(P_{\mathcal{T}}P_{\mathcal{S}}) \leq \|P_{\mathcal{T}}P_{\mathcal{S}}\| + \|(P_{\mathcal{T}}P_{\mathcal{S}})^2\|^{\frac{1}{2}} \leq 2\|P_{\mathcal{T}}P_{\mathcal{S}}\| \leq \|P_{\mathcal{T}} + P_{\mathcal{S}}\|. \tag{3.14}$$

Last inequality can also be obtained from a more general result see [[18], Theorem 1].

From the KKME we can obtain a lower bound for the norm of $P_{\mathcal{T}} - P_{\mathcal{S}}$. Since $1 = \|P_{\mathcal{T}}\| = \|(1 - P_{\mathcal{S}})P_{\mathcal{T}} + P_{\mathcal{S}}P_{\mathcal{T}}\|$ implies that $1 - \|P_{\mathcal{S}}P_{\mathcal{T}}\| \leq \|(1 - P_{\mathcal{S}})P_{\mathcal{T}}\|$. By a similar argument, we can prove that $1 - \|P_{\mathcal{T}}P_{\mathcal{S}}\| \leq \|(1 - P_{\mathcal{T}})P_{\mathcal{S}}\|$. Then

$$1 - \|P_{\mathcal{S}}P_{\mathcal{T}}\| \leq \max\{\|(1 - P_{\mathcal{T}})P_{\mathcal{S}}\|, \|(1 - P_{\mathcal{S}})P_{\mathcal{T}}\|\} = \|P_{\mathcal{T}} - P_{\mathcal{S}}\|. \tag{3.15}$$

We now derive a refinement norm inequality related of (3.15).

Corollary 3.2. *Let $P_{\mathcal{T}}, P_{\mathcal{S}}$ be orthogonal projections on $\mathcal{H}$, then*

$$\begin{aligned} 1 - \|P_{\mathcal{T}}P_{\mathcal{S}}\| \leq 1 - \|P_{\mathcal{T}}P_{\mathcal{S}}\|^2 \leq \|P_{\mathcal{T}} - P_{\mathcal{S}}\|^2 &\leq \|P_{\mathcal{T}} + P_{\mathcal{S}} - 2P_{\mathcal{T}}P_{\mathcal{S}}\| \\ &\leq \|P_{\mathcal{T}} - P_{\mathcal{S}}\|. \end{aligned} \tag{3.16}$$

In particular, $\|P_{\mathcal{T}} - P_{\mathcal{S}}\| = 1 - \|P_{\mathcal{T}}P_{\mathcal{S}}\|$ if and only if either $P_{\mathcal{T}} = P_{\mathcal{S}}$ or $P_{\mathcal{T}}P_{\mathcal{S}} = 0$.

Proof. To prove the second inequality observe that from (3.9), (1.5) and (1.6) we have

$$\|P_{\mathcal{T}}P_{\mathcal{S}}\|^2 + \|P_{\mathcal{T}}P_{\mathcal{S}}\| \geq \frac{1 + 2\|P_{\mathcal{S}}P_{\mathcal{T}}\| + \|P_{\mathcal{S}}P_{\mathcal{T}}\|^2 - \|P_{\mathcal{T}} - P_{\mathcal{S}}\|^2}{2}.$$

Consequently

$$\|P_{\mathcal{T}} - P_{\mathcal{S}}\|^2 \geq 1 - \|P_{\mathcal{S}}P_{\mathcal{T}}\|^2. \tag{3.17}$$

On the other hand, the third and fourth inequalities are consequence of Corollary 2 in [16]. □

It should be remarked here that the inequality (3.16) is sharper than the inequality obtained by Kittaneh in [[14], Corollary 1] when A and B are orthogonal projections.

4. Commutator norm inequality for orthogonal projections

Let $P_{\mathcal{T}}, P_{\mathcal{S}}$ be orthogonal projections on $\mathcal{H}$. It follows by the triangle inequality, the submultiplicativity of the usual operator norm that

$$\|P_{\mathcal{T}}P_{\mathcal{S}} - P_{\mathcal{S}}P_{\mathcal{T}}\| \leq 2\|P_{\mathcal{T}}\|\|P_{\mathcal{S}}\| = 2. \tag{4.1}$$

As $P_{\mathcal{T}}$ or $P_{\mathcal{S}}$ is a positive operator, by Theorem 1 in [16], Kittaneh improved the inequality (4.1) as follows

$$\|P_{\mathcal{T}}P_{\mathcal{S}} - P_{\mathcal{S}}P_{\mathcal{T}}\| \leq \|P_{\mathcal{T}}\|\|P_{\mathcal{S}}\| = 1. \tag{4.2}$$

Furthermore as $0 \leq P_{\mathcal{T}}, P_{\mathcal{S}} \leq I$, in [19] Stampfli refined (4.2) (or see [23]), in the following way

$$\|P_{\mathcal{T}}P_{\mathcal{S}} - P_{\mathcal{S}}P_{\mathcal{T}}\| \leq \frac{1}{2}. \tag{4.3}$$

Moreover, Wang and Du proved that

$$\{\|P_{\mathcal{T}}P_{\mathcal{S}} - P_{\mathcal{S}}P_{\mathcal{T}}\| : P_{\mathcal{T}}, P_{\mathcal{S}} \text{ are orthogonal projections}\} = \left[0, \frac{1}{2}\right]. \tag{4.4}$$

By Corollary 1 in [16], since $P_{\mathcal{T}} + P_{\mathcal{S}}$ is positive, as follows

$$\|P_{\mathcal{T}}P_{\mathcal{S}} - P_{\mathcal{S}}P_{\mathcal{T}}\| \leq \frac{1}{2}\|P_{\mathcal{T}} + P_{\mathcal{S}}\|\|P_{\mathcal{T}} - P_{\mathcal{S}}\|. \tag{4.5}$$

On the other hand, from the anticommutator norm, Walters obtained the following bounds on the commutator norm:

$$\|P_{\mathcal{S}}P_{\mathcal{T}}\| - \|P_{\mathcal{S}}P_{\mathcal{T}}\|^2 \leq \|P_{\mathcal{T}}P_{\mathcal{S}} - P_{\mathcal{S}}P_{\mathcal{T}}\| \leq \|P_{\mathcal{S}}P_{\mathcal{T}}\|. \tag{4.6}$$

Combining the previous inequalities, we have that

$$\begin{aligned}\|P_{\mathcal{S}}P_{\mathcal{T}}\| - \|P_{\mathcal{S}}P_{\mathcal{T}}\|^2 &\leq \|P_{\mathcal{T}}P_{\mathcal{S}} - P_{\mathcal{S}}P_{\mathcal{T}}\| \\ &\leq \min\left\{\frac{1}{2}, \|P_{\mathcal{S}}P_{\mathcal{T}}\|, \frac{1}{2}\|P_{\mathcal{T}} + P_{\mathcal{S}}\|\|P_{\mathcal{T}} - P_{\mathcal{S}}\|\right\}. \end{aligned} \tag{4.7}$$

For an account of the results related to the commutator norm, we invite the reader to consult [19,15,23] and the references therein.

Example 4.1. Observe that $\frac{1}{2}, \|P_{\mathcal{S}}P_{\mathcal{T}}\|$ and $\frac{1}{2}\|P_{\mathcal{T}} + P_{\mathcal{S}}\|\|P_{\mathcal{T}} - P_{\mathcal{S}}\|$ are not comparable as we see in the next example: let $\mathcal{T}, \mathcal{S}$ be two one dimensional subspaces in $\mathbb{R}^2$ be spanned by the vectors u and v, respectively. We choose these vectors such that $\|u\| = 1 = \|v\|$ and $\langle u, v\rangle \geq 0$. Let $w \in \mathbb{R}^2$ be a unit vector orthogonal to u such that $v = u\cos(\theta) + w\sin(\theta)$ with $0 \leq \theta \leq \frac{\pi}{2}$. Hence we get

$$\|P_{\mathcal{T}} - P_{\mathcal{S}}\| = \sin(\theta) \qquad \text{and} \qquad \|P_{\mathcal{S}}P_{\mathcal{T}}\| = \cos(\theta).$$

It follows that $\frac{1}{2}\|P_{\mathcal{T}} + P_{\mathcal{S}}\|\|P_{\mathcal{T}} - P_{\mathcal{S}}\| = \frac{1}{2}(1 + \cos(\theta))\sin(\theta)$.

Then,

(1) There exists $\theta_0 \in [0, \frac{\pi}{2}]$, such that

$$\frac{1}{2}(1 + \cos(\theta_0))\sin(\theta_0) = \frac{1}{2}\|P_{\mathcal{T}} + P_{\mathcal{S}}\|\|P_{\mathcal{T}} - P_{\mathcal{S}}\| < \frac{1}{2} < \|P_{\mathcal{S}}P_{\mathcal{T}}\| = \cos(\theta_0).$$

(2) There exists $\theta_1 \in [0, \frac{\pi}{2}]$, such that $\frac{1}{2} < \frac{1}{2}\|P_{\mathcal{T}} + P_{\mathcal{S}}\|\|P_{\mathcal{T}} - P_{\mathcal{S}}\| < \|P_{\mathcal{S}}P_{\mathcal{T}}\|$.
(3) There exists $\theta_2 \in [0, \frac{\pi}{2}]$, such that $\|P_{\mathcal{S}}P_{\mathcal{T}}\| < \frac{1}{2} < \frac{1}{2}\|P_{\mathcal{T}} + P_{\mathcal{S}}\|\|P_{\mathcal{T}} - P_{\mathcal{S}}\|$.

Remark 4.1.

(1) On the other hand, from the Walter's proof in [21] we have that

$$\|P_{\mathcal{T}}P_{\mathcal{S}} - P_{\mathcal{S}}P_{\mathcal{T}}\| = \|P_{\mathcal{T}}P_{\mathcal{S}}(1 - P_{\mathcal{T}})\| \leq \|(1 - P_{\mathcal{T}})P_{\mathcal{S}}\|. \tag{4.8}$$

Since $(P_{\mathcal{T}}P_{\mathcal{S}} - P_{\mathcal{S}}P_{\mathcal{T}})^* = P_{\mathcal{S}}P_{\mathcal{T}} - P_{\mathcal{T}}P_{\mathcal{S}}$, we have that $\|P_{\mathcal{T}}P_{\mathcal{S}} - P_{\mathcal{S}}P_{\mathcal{T}}\| \leq \|(1 - P_{\mathcal{S}})P_{\mathcal{T}}\|$. From (3.3), we have the following inequality

$$\|P_{\mathcal{T}}P_{\mathcal{S}} - P_{\mathcal{S}}P_{\mathcal{T}}\| \leq \max\{\|(1 - P_{\mathcal{T}})P_{\mathcal{S}}\|, \|(1 - P_{\mathcal{S}})P_{\mathcal{T}}\|\} = \|P_{\mathcal{T}} - P_{\mathcal{S}}\|.$$

In particular, by Corollary 1 in [16] we have

$$\begin{aligned}\|P_{\mathcal{S}}P_{\mathcal{T}}\| - \|P_{\mathcal{S}}P_{\mathcal{T}}\|^2 &\leq \|P_{\mathcal{T}}P_{\mathcal{S}} - P_{\mathcal{S}}P_{\mathcal{T}}\| \\ &\leq \frac{1}{2}\|P_{\mathcal{T}} + P_{\mathcal{S}}\|\|P_{\mathcal{T}} - P_{\mathcal{S}}\| \leq \|P_{\mathcal{T}} - P_{\mathcal{S}}\|.\end{aligned}$$

(2) From the arithmetic-geometric mean inequality we have that

$$\|P_{\mathcal{T}} + P_{\mathcal{S}}\|\|P_{\mathcal{T}} - P_{\mathcal{S}}\| \leq \frac{\|P_{\mathcal{S}} + P_{\mathcal{T}}\|^2 + \|P_{\mathcal{T}} - P_{\mathcal{S}}\|^2}{2}$$

Now we are in a position to present a refinement of the previous inequality. Observe that for any $x \in \mathcal{H}$, we have as consequence of Pythagorean equality that

$$\|P_{\mathcal{T}}x\|^2 + \|P_{\mathcal{S}}x\|^2 = \frac{\|P_{\mathcal{S}}x + P_{\mathcal{T}}x\|^2 + \|P_{\mathcal{T}}x - P_{\mathcal{S}}x\|^2}{2}.$$

Consequently,

$$\langle (P_{\mathcal{T}} + P_{\mathcal{S}})x, x\rangle = \frac{\|P_{\mathcal{S}}x + P_{\mathcal{T}}x\|^2 + \|P_{\mathcal{T}}x - P_{\mathcal{S}}x\|^2}{2}. \tag{4.9}$$

Taking the supremum over $x \in \mathcal{H}, \|x\| = 1$ in (4.9), we have

$$\|P_{\mathcal{T}} + P_{\mathcal{S}}\| \leq \frac{\|P_{\mathcal{S}} + P_{\mathcal{T}}\|^2 + \|P_{\mathcal{T}} - P_{\mathcal{S}}\|^2}{2}. \tag{4.10}$$

Then

$$\|P_{\mathcal{T}} + P_{\mathcal{S}}\|\|P_{\mathcal{T}} - P_{\mathcal{S}}\| \leq \|P_{\mathcal{T}} + P_{\mathcal{S}}\| \leq \frac{\|P_{\mathcal{S}} + P_{\mathcal{T}}\|^2 + \|P_{\mathcal{T}} - P_{\mathcal{S}}\|^2}{2}.$$

Combining the different results we obtain the following statement.

Theorem 4.1. *Let $P_{\mathcal{T}}, P_{\mathcal{S}}$ be orthogonal projections on $\mathcal{H}$, unless $P_{\mathcal{T}} = P_{\mathcal{S}} = 0$, then*
(1)

$$\begin{aligned}\|P_{\mathcal{S}}P_{\mathcal{T}}\| - \|P_{\mathcal{S}}P_{\mathcal{T}}\|^2 &\leq \|P_{\mathcal{T}}P_{\mathcal{S}} - P_{\mathcal{S}}P_{\mathcal{T}}\| \\ &\leq \min\left\{\frac{1}{2}, \|P_{\mathcal{S}}P_{\mathcal{T}}\|, \frac{1}{2}\|P_{\mathcal{T}} + P_{\mathcal{S}}\|\|P_{\mathcal{T}} - P_{\mathcal{S}}\|\right\} \\ &\leq \frac{1}{2}\|P_{\mathcal{T}} + P_{\mathcal{S}}\|\|P_{\mathcal{T}} - P_{\mathcal{S}}\| \\ &\leq \min\left\{\|P_{\mathcal{S}} - P_{\mathcal{T}}\|, \frac{1}{2}\|P_{\mathcal{T}} + P_{\mathcal{S}}\|\right\} \\ &\leq \|P_{\mathcal{S}} - P_{\mathcal{T}}\| \leq 1 \leq 1 + \|P_{\mathcal{S}}P_{\mathcal{T}}\| = \|P_{\mathcal{T}} + P_{\mathcal{S}}\|.\end{aligned}$$

(2)

$$\begin{aligned}\|P_{\mathcal{S}}P_{\mathcal{T}}\| - \|P_{\mathcal{S}}P_{\mathcal{T}}\|^2 &\leq \|P_{\mathcal{T}}P_{\mathcal{S}} - P_{\mathcal{S}}P_{\mathcal{T}}\| \\ &\leq \min\left\{\frac{1}{2}, \|P_{\mathcal{S}}P_{\mathcal{T}}\|, \frac{1}{2}\|P_{\mathcal{T}} + P_{\mathcal{S}}\|\|P_{\mathcal{T}} - P_{\mathcal{S}}\|\right\}\end{aligned}$$

$$\leq \|P_{\mathcal{S}}P_{\mathcal{T}}\| \leq \|P_{\mathcal{S}}P_{\mathcal{T}}\| + \|P_{\mathcal{S}}P_{\mathcal{T}}\|^2 = \|P_{\mathcal{T}}P_{\mathcal{S}} + P_{\mathcal{S}}P_{\mathcal{T}}\|$$
$$\leq 2\omega(P_{\mathcal{S}}P_{\mathcal{T}}) \leq \|P_{\mathcal{T}}P_{\mathcal{S}}\| + \|(P_{\mathcal{T}}P_{\mathcal{S}})^2\|^{\frac{1}{2}}$$
$$\leq 2\|P_{\mathcal{S}}P_{\mathcal{T}}\| \leq 1 + \|P_{\mathcal{S}}P_{\mathcal{T}}\| = \|P_{\mathcal{T}} + P_{\mathcal{S}}\|.$$

An application of Theorem 4.1 yields necessary and sufficient conditions for the equality in (1.3). In [[9], Theorem 1.3-4], the authors proved that if $R(T) \perp R(T^*)$, then $\omega(T) = \frac{1}{2}\|T\|$. We show that for the product of two orthogonal projections such condition is also necessary.

Corollary 4.1. *Let $P_{\mathcal{T}}, P_{\mathcal{S}}$ be orthogonal projections on $\mathcal{H}$. Then*

(1) $\omega(P_{\mathcal{S}}P_{\mathcal{T}}) = \frac{1}{2}\|P_{\mathcal{S}}P_{\mathcal{T}}\|$ *if and only if* $P_{\mathcal{S}}P_{\mathcal{T}} = 0$.
(2) $\omega(P_{\mathcal{S}}P_{\mathcal{T}}) = \|P_{\mathcal{S}}P_{\mathcal{T}}\|$ *if and only if either* $P_{\mathcal{S}}P_{\mathcal{T}} = 0$ *or* $\|P_{\mathcal{S}}P_{\mathcal{T}}\| = 1$.

The next proposition is an example of localization of the numerical radius of a product of two orthogonal projections using the previous inequalities obtained.

Proposition 4.1. *Let $P_{\mathcal{T}}, P_{\mathcal{S}}$ be orthogonal projections on $\mathcal{H}$ Then*

$$\left|\omega(P_{\mathcal{S}}P_{\mathcal{T}}) - \left(\frac{1 + \|P_{\mathcal{S}}P_{\mathcal{T}}\|}{2}\right)^2\right| \leq \frac{1}{4}\|P_{\mathcal{T}} - P_{\mathcal{S}}\|^2. \tag{4.11}$$

Proof. The inequalities (3.13), (3.14) together with (4.10), imply that

$$\frac{\|P_{\mathcal{S}} + P_{\mathcal{T}}\|^2 - \|P_{\mathcal{T}} - P_{\mathcal{S}}\|^2}{2} \leq 2\omega(P_{\mathcal{T}}P_{\mathcal{S}}) \leq \frac{\|P_{\mathcal{S}} + P_{\mathcal{T}}\|^2 + \|P_{\mathcal{T}} - P_{\mathcal{S}}\|^2}{2} \quad \square \tag{4.12}$$

Taking up the Example 4.1 once, we have that for any $\theta \in [0, \frac{\pi}{2}]$ holds that

$$\omega(P_{\mathcal{S}}P_{\mathcal{T}}) \leq \frac{\|P_{\mathcal{S}} + P_{\mathcal{T}}\|^2 + \|P_{\mathcal{T}} - P_{\mathcal{S}}\|^2}{4} = \frac{1 + \cos(\theta)}{2} < \frac{\sqrt{17}}{4}.$$

Now we have this simple consequence of Proposition 4.1.

Corollary 4.2. *Let $P_{\mathcal{T}}, P_{\mathcal{S}}$ be orthogonal projections on $\mathcal{H}$ such that $P_{\mathcal{S}}P_{\mathcal{T}} = 0$, unless $P_{\mathcal{T}} = P_{\mathcal{S}} = 0$, then $\|P_{\mathcal{T}} - P_{\mathcal{S}}\| = 1$.*

Proof. To prove the equality, note that from (4.11) and Corollary 4.1 we have

$$\frac{1}{4} \leq \frac{1}{4}\|P_{\mathcal{T}} - P_{\mathcal{S}}\|^2 \leq \frac{1}{4},$$

which proves that $\|P_{\mathcal{T}} - P_{\mathcal{S}}\| = 1$. $\square$

Proposition 4.2. *Let $P_{\mathcal{T}}, P_{\mathcal{S}}$ be orthogonal projections with $P_{\mathcal{T}} \neq 0$ and $P_{\mathcal{S}} \neq 0$. Then, the following conditions are equivalent:*

(1) $P_{\mathcal{T}} + P_{\mathcal{S}}$ *is an orthogonal projection.*
(2) $P_{\mathcal{T}}P_{\mathcal{S}} = 0$.
(3) $\|P_{\mathcal{T}} + P_{\mathcal{S}}\| = 1$.

Proof. Suppose $P_{\mathcal{T}} + P_{\mathcal{S}}$ is an orthogonal projection. Then, $0 = \|P_{\mathcal{T}}P_{\mathcal{S}} + P_{\mathcal{S}}P_{\mathcal{T}}\| = \|P_{\mathcal{T}}P_{\mathcal{S}}\|^2 + \|P_{\mathcal{T}}P_{\mathcal{S}}\|$ and thus $P_{\mathcal{T}}P_{\mathcal{S}} = 0$.

Now if $P_{\mathcal{T}}P_{\mathcal{S}} = 0$ so it is immediately consequence of (1.5) that $\|P_{\mathcal{T}} + P_{\mathcal{S}}\| = 1$. Finally if $\|P_{\mathcal{T}} + P_{\mathcal{S}}\| = 1$, then Choi and Wu in [[5], Proposition 2.1] showed that $P_{\mathcal{T}} + P_{\mathcal{S}}$ is an orthogonal projection. □

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