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On the spectral radius of block graphs having all their blocks of the same size

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A B S T R A C T

Let $\mathcal{B}(n,q)$ be the class of block graphs on n vertices having all their blocks of the same size. We prove that if $G \in \mathcal{B}(n,q)$ has at most three pairwise adjacent cut vertices then the minimum spectral radius $\rho(G)$ is attained at a unique graph. In addition, we present a lower bound for $\rho(G)$ when $G \in \mathcal{B}(n,q)$.

1. Introduction

The problem of finding those graphs that maximize or minimize the spectral radius of a connected graph on n vertices, within a given graph class $\mathcal{H}$, has attracted the attention of many researchers. Usually, this kind of problem is solved through graph transformations

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preserving the number of vertices, so that the resulting graph also belongs to $\mathcal{H}$, and has a monotone behavior with respect to the spectral radius. We refer the reader to [12] for more details about this technique and others. In [9], Lovász and Pelikán proved that the unique graph with maximum spectral radius among trees on n vertices is the star $K_{1,n-1}$, and that the unique graph with minimum spectral radius is the path P_n. As far as we know, this article is the first one within this research line. Since adding edges to a graph increases the spectral radius (see Corollary 1), if $\mathcal{H}$ contains complete graphs and paths, then K_n maximizes and P_n minimizes $\rho(G)$ among graphs in $\mathcal{H}$, meaning that these two graphs have the minimum and the maximum spectral radius among graphs on n vertices when $\mathcal{H}$ is the class of all connected graphs. Consequently, several authors have considered the problem when $\mathcal{H}$ is a graph class not containing either paths or complete graphs and defined by a restriction of classical graph parameters. Graphs with a given independence number [10,5], and graphs with a given clique number [13] and graphs with given connectivity and edge-connectivity [6]. It is worth mentioning that the foundation stone that gives place to many subsequent articles in connection with this problem is that of Brualdi and Solheid [3]. For concepts and definitions used in this section, we refer the reader to Section 2.

In this article, we consider the class $\mathcal{B}(n,q)$ of block graphs on n vertices having all their blocks on $q+1$ vertices, for every $q \geq 2$. For results related to the adjacency matrix of block graphs we refer to [2]. Trees are block graphs with all their blocks on two vertices. In connection with the spectral radius on trees, the following result was obtained.

Theorem 1.1. *[9] If T is a tree on n vertices, then $2\cos\left(\frac{\pi}{n+1}\right) = \rho(P_n) \leq \rho(T) \leq \rho(K_{1,n-1}) = \sqrt{n-1}$.*

In an attempt to generalize Theorem 1.1, we find the unique graph in $G \in \mathcal{B}(n,q)$ that reaches the minimum spectral $\rho(G)$ in the case in which G has at most three pairwise adjacent cut vertices. Besides, we present a lower bound for $\rho(G)$.

Theorem 1.2. *If $G \in \mathcal{B}(n,q)$, then $\rho(G) \leq \rho(S(n,q))$ and $S(n,q)$ is the unique graph that maximizes the spectral radius. In addition, if G has at most three pairwise adjacent cut vertices then $\rho\left(P_b^q\right) \leq \rho(G)$ and P_b^q is the unique graph that minimizes the spectral radius in the class $\mathcal{B}(n,q)$, where $b = \frac{n-1}{q}$.*

The upper bound was already obtained in [8], using a notion of adjacency matrix for uniform hypertrees equivalent to the adjacency matrix of blocks graphs having all their blocks of the same size. To the best of our knowledge, there is no result about the lower bound. Many articles have been published on uniform hypertrees but using a tensor associated to them, but this concept has no immediate connection with the notion of adjacency matrix presented in [8]. We have strong evidence, obtained by the aid of Sage software, that the hypothesis of having at most three pairwise adjacent cut vertices; in connection with the minimum of the spectral radius; can be dropped.

This article is organized as follows. In Section 2 we present some definitions and preliminary results. Section 3 presents two graph transformations having a monotone behavior with respect to the spectral radius. Section 4 is devoted to putting together all previous results, to prove our main result. Section 5 presents a lower bound for the spectral radius. Finally, Section 6 contains a summary of our work and two conjectures are posted.

2. Preliminaries

2.1. Definitions

All graphs; mentioned in this article, are finite, and have neither loops nor multiple edges. Let G be a graph. We use $V(G)$ and $E(G)$ to indicate the set of vertices and the set of edges of G, respectively. A graph on one vertex is called a *trivial graph*. Let v be a vertex of G, $N_G(v)$ (resp. $N_G[v]$) stands for the (closed) neighborhood of v (resp. $N_G(v) \cup \{v\}$), if the context is clear the subscript G is omitted. We use $d_G(v)$ to symbolize the degree of v in G, or $d(v)$, provided the context is clear. By $\overline{G}$ we denote the complement graph of G. Given a set F of edges of G (resp. of $\overline{G}$), we mean by $G - F$ (resp. $G + F$) the graph obtained from G by removing (resp. adding) all the edges in F. If $F = \{e\}$, we use $G - e$ (resp. $G + e$) for short. Let $X \subseteq V(G)$, we use $G[X]$ to indicate the graph induced by X. By $G - X$ we denote the graph $G[V(G) \setminus X]$. If $X = \{v\}$, we use $G - v$ for short. Let G and H be two graphs, we use $G + H$ to designate the disjoint union between G and H, and G^+ stands for the graph obtained by adding an isolated vertex to G. We mean by P_n and K_n the path and the complete graph on n vertices.

We mean by $A(G)$ the adjacency matrix of G, and $\rho(G)$ stands for the spectral radius of $A(G)$, we refer to $\rho(G)$ as the spectral radius of G. If x is the principal eigenvector of $A(G)$, which is indexed by $V(G)$, we use x_u to indicate the coordinate of x corresponding to the vertex u. We use $P_G(x)$ to designate the characteristic polynomial of $A(G)$; i.e., $P_G(x) = \det(xI_n - A(G))$. It is easy to prove that $P_{K_n}(x) = (x - n + 1)(x + 1)^{n-1}$.

A vertex v of a graph G is a *cut vertex* if $G - v$ has a number of connected components greater than the number of connected components of G. Let H be a graph. A *block* of H, also known as a 2-*connected component*, is a maximal connected subgraph of H having no cut vertex. A *block graph* is a connected graph whose blocks are complete graphs. We use $\mathcal{B}(n, q)$ to denote the family of block graphs on n vertices whose blocks have $q + 1$ vertices. Notice that if $B \in \mathcal{B}(n, q)$ and b is its number of blocks, then $b = \frac{n-1}{q}$. Let G be a block graph, a *leaf block* is a block of G such that it contains exactly one cut vertex of G. We use $S(n, q)$ to designate the block graph in $\mathcal{B}(n, q)$ having b blocks with only one cut vertex. By P_b^q we designate the block graph in $\mathcal{B}(bq + 1, q)$ with at most two leaf blocks when $n - 1 > q$ and no cut vertices when $n - 1 = q$, called (q, b)-path-block.

2.2. Preliminary results

This subsection is split into two parts. In the first one, we present the results needed to deal with the minimum spectral radius in $\mathcal{B}(n, q)$, and in the second one, we briefly describe the previous result in connection with the maximum spectral radius in this class.

Tools for the minimum. We will introduce a partial order on the class of graphs. We will use it to develop graph transformations used to prove our main result. This technique was pioneered by Lovász and Pelikán [9].

Definition 1. Let G and H be two graphs. We mean by $G \prec H$, if $P_H(x) > P_G(x)$ for all $x \geq \rho(G)$.

It is immediate that if $G \prec H$ then $\rho(H) < \rho(G)$. The spectrum radius is nondecreasing with respect to the subgraph partial order.

We repeatedly use the following Lemma to deal with the subgraph partial order previously defined.

Lemma 2.1. *If H is a proper subgraph of G, then $\rho(H) < \rho(G)$.*

The reader is referred to [1] for the proof of the above lemma. In particular, adding edges to a graph increases the spectral radius.

Corollary 1. *If G is a graph such that $uv \notin E(G)$, then $\rho(G) < \rho(G + uv)$.*

The following technical lemma is a useful tool to develop graph transformations.

Lemma 2.2. *[7] If H is a spanning subgraph of the graph G, then $P_G(x) \leq P_H(x)$ for all $x \geq \rho(G)$. In addition, if G is connected, then $G \prec H$.*

Let G and H be two graphs. If $g \in V(G)$ and $h \in V(H)$, the *coalescence* between G and H at g and h, denoted $G \cdot_g^h H$, is the graph obtained from G and H, by identifying vertices g and h (see Fig. 1). We use $G \cdot H$ for short. Notice that any block graph can be constructed by recursively using the coalescence operation between a block graph and a complete graph.

In the 70s, Schwenk published an article containing useful formulas for the characteristic polynomial of a graph [11]. The part corresponding to minimizing the spectral radius of the main result of this research is based on the following Schwenk's formula, linking the characteristic polynomial of two graphs and the coalescence between them.

Lemma 2.3. *[11] Let G and H be two graphs. If $g \in V(G)$, $h \in V(H)$, and $F = G \cdot H$, then*

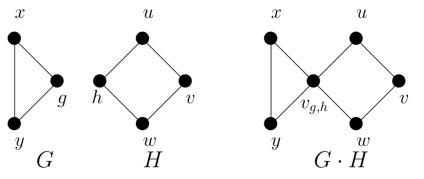

Fig. 1. The coalescence of graphs G and H at vertices g and h.

$$P_F(x) = P_G(x)P_{H-h}(x) + P_{G-g}(x)P_H(x) - xP_{G-g}(x)P_{H-h}(x).$$

More details on Lemmas 2.2 and 2.3 can be found in [4].

The following two technical lemmas will play an important role to prove the main result of this article.

Lemma 2.4. *Let H be a graph, let $v, w \in V(H)$ such that $H - w \prec H - v$, and let G be a connected graph. If G_1 and G_2 are the graphs obtained by means of the coalescence between G and H at $u \in V(G)$, and v or w respectively, then $G_1 \prec G_2$.*

Proof. By Lemma 2.3, the characteristic polynomials of G_1 and G_2 are

$$P_{G_1}(x) = P_{G-u}(x)P_H(x) + (P_G(x) - xP_{G-u}(x))P_{H-v}(x)$$

and

$$P_{G_2}(x) = P_{G-u}(x)P_H(x) + (P_G(x) - xP_{G-u}(x))P_{H-w}(x),$$

respectively, and thus

$$P_{G_2}(x) - P_{G_1}(x) = (P_G(x) - xP_{G-u}(x))(P_{H-w}(x) - P_{H-v}(x)). \tag{1}$$

By Lemmas 2.1 and 2.2, $G \prec (G-u)^+$. Therefore, since $H - w \prec H - v$, by Lemma 2.1 and (1), we have $G_1 \prec G_2$. $\square$

Lemma 2.5. *Let H_1, H_2 be two graphs such that either $H_1 = H_2$ or $H_1 \prec H_2$, let $v_i \in V(H_i)$ for each $i = 1, 2$ such that $H_2 - v_2 \prec H_1 - v_1$, and let G be a connected graph. If G_i is the graph obtained by means of the coalescence between G and H_i at $v \in V(G)$ and v_i for each $i = 1, 2$, then $G_1 \prec G_2$.*

Proof. By applying Lemma 2.3 as in Lemma 2.4, we obtain

$$\begin{aligned} P_{G_2}(x) - P_{G_1}(x) = (P_G(x) - xP_{G-v}(x))(P_{H_2-v_2}(x) - P_{H_1-v_1}(x)) \\ + (P_{H_2}(x) - P_{H_1}(x))P_{G-v}(x). \end{aligned} \tag{2}$$

By Lemmas 2.1 and 2.2, $G \prec (G-v)^+$. Therefore, since either $H_1 = H_2$ or $H_1 \prec H_2$ and $H_2 - v_2 \prec H_1 - v_1$, by (2) and Lemma 2.1, we conclude that $G_2 \prec G_1$. $\square$

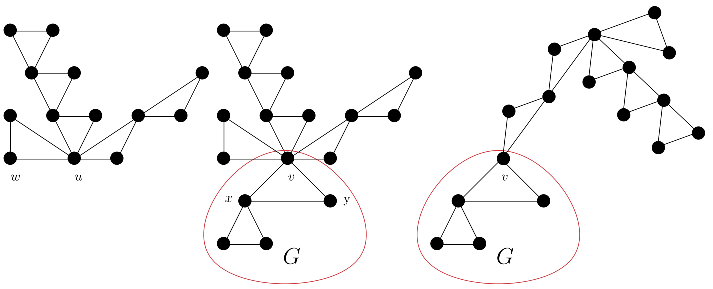

Fig. 2. From left to right H, H_1, and H_2.

Tools for finding the maximum. The below theorem was proved in the context of studying the spectral radius of the adjacency matrix of the $q+1$-uniform hypertrees. This matrix agrees with that of a block graph having all their blocks of size $q+1$.

Theorem 2.1. *[8, Theorem 4.1] If $G \in \mathcal{B}(n,q)$, then*

$$\rho(G) \leq \rho(S(n,q)) = \frac{q-1+\sqrt{(q-1)^2+4(n-1)}}{2}.$$

Besides, $S(n,q)$ is the unique graph maximizing $\rho(G)$.

Although the minimum spectral radius of a $q+1$-uniform hypertrees was also considered in the literature under the notion of spectral radius of the hypermatrix, also called tensor, associated to a uniform hypertree (see e.g. [14, Corollary 19]), the reader should notice that this notion does not have an immediate relationship with the notion of the spectral radius of the adjacency matrix of a block graph.

3. Graph transformations

To ease the reading of the next proposition we recommend seeing Fig. 2.

Proposition 1. *Let G be a connected graph, and let $u \in V(G)$ such that $G-u$ is connected. Let H be the graph obtained from $S(k(q-1)+1,k)$ by adding for all $1 \leq i \leq k$ one pendant (q,b_i)-path-block (possibly empty, i.e., $b_i = 0$) to each leaf block, let $v \in V(H)$ be the vertex of degree $k(q-1)$ and let $w \in V(H)$ be any vertex in leaf block of H. If H_1 is the graph obtained by the coalescence between G and H at u and v, H_2 is the graph obtained by the coalescence between G and H at u and w, then $H_1 \prec H_2$.*

Proof. Observe that $H-w$ is connected and $H-v$ is a disconnected spanning subgraph of $H-w$. Thus, by Lemma 2.2 $H-w \prec H-v$. Therefore, the result follows from Lemma 2.4. □

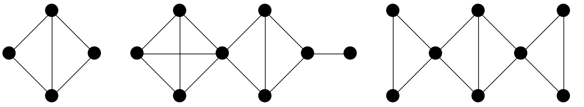

Fig. 3. From left to right G, $G(4,1)$ and $G(3,3)$.

The following proposition plays a central role to prove the main result of this article. We use $G(r,s)$ to indicate the graph obtained by means of the coalescence between G and a copy of K_r at $u \in V(G)$ and any vertex of the complete graph, and between G and K_s at $v \in V(G)$ and any vertex of the complete graph (see the example depicted in Fig. 3).

Proposition 2. *Let G be a connected graph and let $u, v \in V(G)$. If r and s are two integers such that $1 \le r \le s-2$, $G-u \prec G-v$ or $G-u = G-v$, then $G(r,s) \prec G(r+1,s-1)$.*

Proof. By applying Lemma 2.3 to $G(r,s)$ at v, we obtain

$$P_{G(r,s)}(x) = (x+1)^{s-2}[(x-s+2)P_{G(r,s)-K_{s-1}}(x) - (s-1)P_{G(r,s)-K_s}(x)]. \quad (3)$$

Applying again Lemma 2.3 to $P_{G(r,s)-K_{s-1}}(x)$ and $P_{G(r,s)-K_s}(x)$, we obtain

$$\begin{aligned} P_{G(r,s)}(x) = (x+1)^{s+r-4}\{&(x-s+2)[(x-r+2)P_G(x) - (r-1)P_{G-u}(x)] \\ &- (s-1)[(x-r+2)P_{G-v}(x) - (r-1)P_{G-\{u,v\}}(x)]\}. \end{aligned}$$

By symmetry

$$\begin{aligned} P_{G(r+1,s-1)}(x) = (x+1)^{s+r-4}\{&(x-s+3)[(x-r+1)P_G(x) - rP_{G-u}(x)] \\ &- (s-2)[(x-r+1)P_{G-v}(x) - rP_{G-\{u,v\}}(x)]\}. \end{aligned}$$

Hence

$$\begin{aligned} P_{G(r+1,s-1)}(x) - P_{G(r,s)}(x) = (x+1)^{s+r-4}\{&(s-r-1)[P_G(x) + P_{G-u}(x) \\ &+ P_{G-v}(x) + P_{G-\{u,v\}}(x)] + (x+1)(P_{G-v}(x) \\ &- P_{G-u}(x))\}. \end{aligned} \quad (4)$$

Therefore, if $1 \le r \le s-2$, by (4) and Lemma 2.1, $G(r,s) \prec G(r+1,s-1)$. □

A (q,b)-path-block in $\mathcal{B}(n,q)$ have blocks $B_1, \ldots, B_b$ such that $V(B_i) \cap V(B_{i+1}) = \{v_i\}$ for every $1 \le i \le b-1$ and $V(B_i) \cap V(B_j) = \emptyset$ whenever $1 \le i < j \le n$ and $|i-j| > 1$. Let G be a graph and let $v, w \in V(G)$ be two adjacent vertices. We use $G[q,k,\ell]$ to mean the graph obtained by adding a pendant (q,ℓ)-path-block at v and a pendant (q,k)-path-block at w, where $1 \le \ell \le k$, and $G[q,r,0]$ stands for the graph obtained from G by adding just a (q,r)-path block at w. By "pendant at v", we mean identifying a noncut vertex from one of the two leaf blocks with a noncut vertex $v \in V(B_1)$ (see Fig. 4).

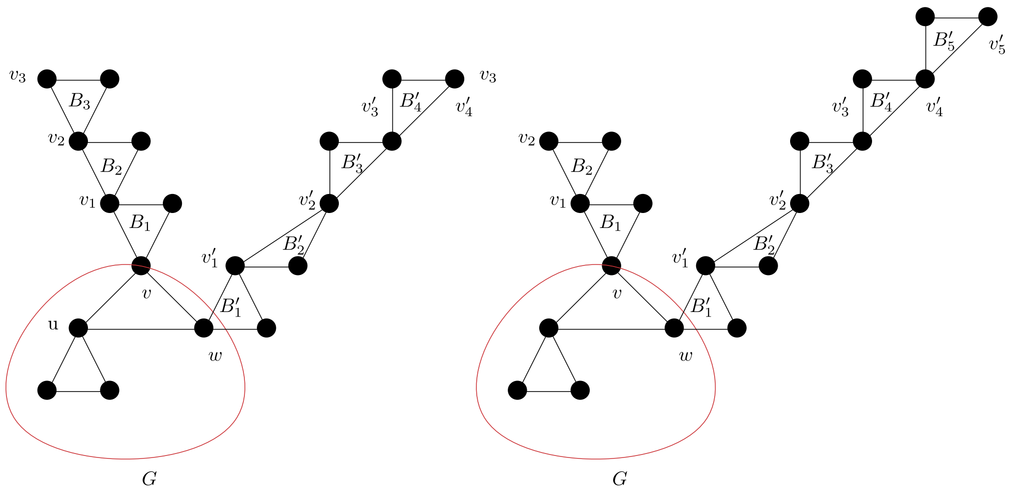

Fig. 4. From left to right $G[3,3,4]$ and $G[3,2,5]$.

Proposition 3. *Let $G \in \mathcal{B}(n,q)$ with at least one cut vertex. If ℓ and k are two positive integers such that $1 \leq \ell \leq k$ and $G[q,k,\ell]$ has at most three adjacent cut vertices, then $G[q,k,\ell] \prec G[q,k+1,\ell-1]$.*

Proof. We mean by P_ℓ^q (resp. P_k^q) the (q,ℓ)-path-block (resp. (q,k)-path-block) whose blocks are $B_1,\dots,B_\ell$ (resp. $B_1',\dots,B_k'$), where $v=v_0$, $w=v_0'$, and v_ℓ and v_k' stand for a fixed arbitrary noncut vertex of B_ℓ and B_k', respectively. By $G^l[q,k,\ell]$ (resp. $G^r[q,k,\ell]$) we indicate the graph $G[q,k,\ell]-v_\ell$ (resp. $G[q,k,\ell]-v_k'$). When both vertices are removed, we use $G^{lr}[q,k,\ell]$. By applying Lemma 2.3 to $G^l[q,k,\ell-1]$ at $v_{\ell-1}$ the following identity is derived.

$$P_{G[q,k,\ell]}(x) = (x+1)^{q-1}((x-q+1)P_{G[q,k,\ell-1]}(x) - qP_{G^l[q,k,\ell-1]}(x)). \tag{5}$$

Analogously

$$P_{G[q,k,\ell]}(x) = (x+1)^{q-1}((x-q+1)P_{G[q,k-1,\ell]}(x) - qP_{G^r[q,k-1,\ell]}(x)). \tag{6}$$

By combining (5) and (6) we obtain

$$P_{G[q,k+1,\ell-1]}(x) - P_{G[q,k,\ell]}(x) = q(x+1)^{q-1}(P_{G^l[q,k,\ell-1]}(x) - P_{G^r[q,k,\ell-1]}(x))). \tag{7}$$

Again, by using properly Lemma 2.3, we derive the next identity

$$\begin{aligned} P_{G^l[q,k,\ell]}(x) = (x+1)^{2q-3}\{&(x-q+1)[(x-q+2)P_{G[q,k-1,\ell-1]}(x) \\ &- (q-1)P_{G^l[q,k-1,\ell-1]}(x)] + q((q-1)P_{G^{lr}[q,k-1,\ell-1]}(x) \\ &- (x-q+2)P_{G^r[q,k-1,\ell-1]}(x))\}. \end{aligned}$$

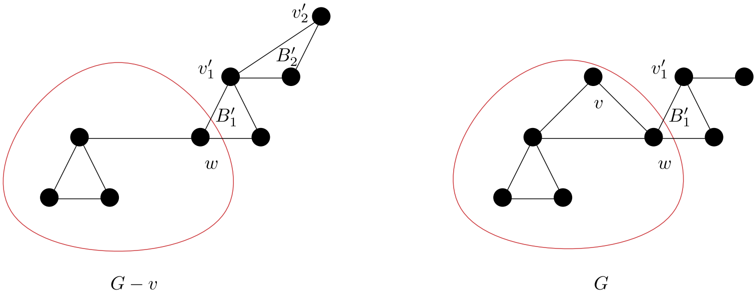

Fig. 5. From left to right $(G-v)[3,2,0]$ and $G^r[3,2,0]$.

Analogously,

$$P_{G^r[q,k,\ell]}(x) = (x+1)^{2q-3}\{(x-q+1)[(x-q+2)P_{G[q,k-1,\ell-1]}(x) \\ -(q-1)P_{G^r[q,k-1,\ell-1]}(x)] + q((q-1)P_{G^{lr}[q,k-1,\ell-1]}(x) \\ -(x-q+2)P_{G^l[q,k-1,\ell-1]}(x))\}.$$

Hence

$$P_{G^l[q,k,\ell]}(x) - P_{G^r[q,k,\ell]}(x) = (x+1)^{2(q-1)}(P_{G^\ell[q,k-1,\ell-1]}(x) \\ - P_{G^r[q,k-1,\ell-1]}(x)). \tag{8}$$

By applying (8) repeatedly, we obtain for every $1 \le j \le \ell$

$$P_{G^l[q,k,\ell]}(x) - P_{G^r[q,k,\ell]}(x) = (x+1)^{2j(q-1)}(P_{G^l[q,k-j,\ell-j]}(x) \\ - P_{G^r[q,k-j,\ell-j]}(x)). \tag{9}$$

Replacing in (7), we obtain that for every $0 \le j \le \ell-1$

$$P_{G[q,k+1,\ell-1]}(x) - P_{G[q,k,\ell]}(x) = q(x+1)^{(2j+1)(q-1)}(P_{G^l[q,k-j,\ell-j-1]}(x) \\ - P_{G^r[q,k-j,\ell-j-1]}(x)). \tag{10}$$

In particular, if $j = \ell-1$

$$P_{G[q,k+1,\ell-1]}(x) - P_{G[q,k,\ell]}(x) = q(x+1)^{(2\ell-1)(q-1)}(P_{(G-v)[q,k-\ell+1,0]}(x) \\ - P_{G^r[q,k-\ell+1,0]}(x)). \tag{11}$$

Thus, it suffices to prove that $G^r[q,t,0] \prec (G-v)[q,t,0]$ for all $t \ge 1$ (see Fig. 5). First observe that, since G has at least one cut vertex, there exists a graph $H_1 \in \mathcal{B}(n,q)$ such that $G^r[q,t,0]$ is the coalescence between H_1 and $P^q_{t+1} - v_{t+1}$ at a noncut vertex $x \in V(H_1)$ and a noncut vertex $v_0 \in V(B_1)$ of $P^q_{t+1} - v_{t+1}$ respectively. Analogously, $(G-v)[q,t,0]$ is the coalescence between H_1 and $P^q_{t+1} - v_0$ at a noncut vertex $x \in V(H_1)$ and a noncut vertex $y \in V(B_1)\setminus\{v_0\}$ of $P^q_{t+1} - v_0$ respectively. From this observation combined with Lemma 2.5 and Proposition 2, we conclude that $G^r[q,t,0] \prec (G-v)[q,t,0]$. □

4. Main result

Let $G \in \mathcal{B}(n,q)$ and let B be a block of G. We say that B is a *special block of type one* if B has at least two pendant path-blocks at $v \in V(B)$ (see Fig. 2, the block whose vertex set is $\{v,x,y\}$ is a special block of type one). We say that B is a *special block of type two* if B has a pendant path-block at $v \in V(B)$ and a pendant path-block at $w \in V(B)$ with $v \neq w$ (see Fig. 4, the block whose vertex set is $\{u,v,w\}$ is a special block of type two). The following lemma, whose proof is omitted, will be used to prove our main result.

Lemma 4.1. *If $G \in \mathcal{B}(n,q)$, then G either is a (q,b)-path-block, has a special block of type one, or has a special block of type two.*

Now we are ready to put all pieces together to prove the main result of the article.

Proof of Theorem 1.2. The upper bound follows from Theorem 2.1. Assume that $G \in \mathcal{B}(n,q)$ and it is not a path-block. By Lemma 4.1, G has either a special block of type one or a special block of type two. Hence, by Propositions 1 and 3, there exists a graph transformation onto G, involving the corresponding pendant path-blocks, such that the resulting graph G' satisfies $\rho(G') < \rho(G)$ and G' has a special block less than G. Therefore, continuing with this procedure as long as G' is a path-block, we conclude that $\rho\left(P^q_{\frac{n-1}{q}}\right) < \rho(G)$ for all $G \in \mathcal{B}(n,q)$. □

5. Lower bound for the spectral radius

Theorem 5.1. *Let $G \in \mathcal{B}(n,q)$ and let $n-1 > q$. Then,*

$$\rho(G) \geq q + \frac{\sqrt{q}}{2}$$

for every $2 \leq q \leq 4$, and

$$\rho(G) \geq q + \frac{4+(q-1)\sqrt{2}}{q+3\sqrt{2}},$$

for every $q \geq 5$.

Proof. Assume now that $q \geq 2$ and $b \geq 3$. By Lemma 2.1, we know that $\rho(P^q_3) \leq \rho(P^q_b) \leq \rho(G)$ for every graph $G \in \mathcal{B}(n,q)$. By simple calculation, using Lemma 2.3, we obtain the characteristic polynomial of P^q_3

$$P_{P_3^q}(x) = (x+1)^{3q-4}\Big((x-q)(x+2)+1\Big)f_q(x), \tag{12}$$

where $f_q(x) := (x-q)\Big((x-q)(x+2)+1\Big)-2q$. Since $f_q(x) = -2q$ when $(x-q)(x+2)+1 = 0$, we have that $\rho(P_3^q)$ is the greatest root of $f_q(x)$. Furthermore, since $f_q(x)$ is an increasing function on $(q, +\infty)$ and $f_q(q) < 0$, we have that $\rho(P_3^q)$ is the unique root of $f_q(x)$ on $(q, +\infty)$.

On one hand,

$$f_q\left(q+\frac{\sqrt{q}}{2}\right) = \frac{q+4}{8}\sqrt{q} + \frac{q(q-6)}{4} \leq 0 \qquad \text{and} \qquad f_q(q+1) = 4-q \geq 0,$$

for every $2 \leq q \leq 4$. Hence

$$\rho(G) \geq q + \frac{\sqrt{q}}{2}.$$

On the other hand,

$$f_q(q+1) = 4-q \leq 0 \qquad \text{and} \qquad f_q(q+\sqrt{2}) = 4+3\sqrt{2} \geq 0,$$

for every $q \geq 5$. Taking into account that $f_q''(x) > 0$ for every $x \in (q, +\infty)$ we conclude that

$$\rho(G) \geq q + \frac{4+(q-1)\sqrt{2}}{q+3\sqrt{2}},$$

where lower bound is the root of the linear function passing through the points $(q+1, 4-q)$ and $(q+\sqrt{2}, 4+3\sqrt{2})$. □

6. Discussions and further research

We have presented three graph transformations to deal with the minimum spectral radius of this class of block graphs, namely Propositions 1, 2, and 3, but the last one has a very strong hypothesis on graph G. We do not know if they can be weakened. Nevertheless, we have collected very strong computational evidence that drives us to the following conjecture.

Conjecture 1. *If $G \in \mathcal{B}(n,q) \setminus \{P_b^q\}$, then $G \prec P_b^q$.*

Consequently, if this statement were true, the following weaker conjecture would be also true.

Conjecture 2. *If $G \in \mathcal{B}(n,q) \setminus \{P_b^q\}$, then $\rho(P_b^q) < \rho(G)$.*

We believe that for proving Conjecture 1 new graph transformations need to be developed.

Another interesting graph class to study the problem of finding the maximum and minimum spectral radius, related to the one considered in this paper, is the class formed by those block graphs on n vertices having exactly b blocks not necessary all of them with the same size.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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