Linear Algebra and its Applications 498 (2016) 441–449

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Norm inequalities related to p-Schatten class

Cristian Conde [a,b], Mohammad Sal Moslehian [c,*]

[a] Instituto de Ciencias, Univ. Nac. Gral. Sarmiento, J.M. Gutierrez 1150, Los Polvorines, Argentina
[b] Instituto Argentino de Matemática "Alberto P. Calderón" – CONICET, Saavedra 15, 3 Piso, (1083) Ciudad Autónoma de Buenos Aires, Argentina
[c] Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran

A R T I C L E   I N F O

Article history:
Received 28 January 2015
Accepted 15 November 2015
Available online 17 December 2015
Submitted by R. Brualdi

Dedicated to the late Prof. Hans Schneider for his outstanding contributions to Linear Algebra

MSC:
47A30
46C15
47B10
47B15

Keywords:
Schatten p-norm
Positive operator
Inequality
Inner product space

A B S T R A C T

In this paper, we obtain some refinements of the known operator inequalities for the p-Schatten class. In addition, we obtain an approach to the inequalities conjectured by Audenaert and Kittaneh for the p-Schatten class.

* Corresponding author.
E-mail addresses: cconde@ungs.edu.ar (C. Conde), moslehian@um.ac.ir, moslehian@member.ams.org (M.S. Moslehian).

http://dx.doi.org/10.1016/j.laa.2015.11.031
0024-3795/

1. Introduction

Let $\mathbb{B}(\mathscr{H})$ denote the algebra of all bounded linear operators acting on a complex separable Hilbert space $\mathscr{H}$. If $X \in \mathbb{B}(\mathscr{H})$ is compact, we denote by $\{s_j(X)\}$ the sequence of singular values of X, i.e. the eigenvalues of $|X| = (X^*X)^{\frac{1}{2}}$, in decreasing order and repeated according to multiplicity. For $p > 0$, let $\|X\|_p = (\sum_j s_j(X)^p)^{1/p} = (\mathrm{tr}\,|X|^p)^{1/p}$, where tr is the usual trace functional. This defines a norm (quasi-norm, resp.) for $1 \le p < \infty$ $(0 < p < 1$, resp.) on the set

$$\mathbb{B}_p(\mathscr{H}) = \{X \in \mathbb{B}(\mathscr{H}) : \|X\|_p < \infty\},$$

which is called the p-Schatten class of $\mathbb{B}(\mathscr{H})$; cf. [5].

Clarkson's inequalities for operators in $\mathbb{B}_p(\mathscr{H})$ (see [13]) assert that for $0 < p \le 2$

$$2^{p-1}(\|A\|_p^p + \|B\|_p^p) \le \|A - B\|_p^p + \|A + B\|_p^p \le 2(\|A\|_p^p + \|B\|_p^p), \tag{1.1}$$

and for $2 \le p < \infty$

$$2(\|A\|_p^p + \|B\|_p^p) \le \|A - B\|_p^p + \|A + B\|_p^p \le 2^{p-1}(\|A\|_p^p + \|B\|_p^p). \tag{1.2}$$

For $p = 2$ both inequalities (1.1) and (1.2) reduce to the *parallelogram law*

$$\|A - B\|_2^2 + \|A + B\|_2^2 = 2(\|A\|_2^2 + \|B\|_2^2).$$

This equality is related to the characterization of inner product spaces due to Jordan and von Neumann [12] in the following sense: let E be a real normed linear space. Then E is an inner product space if and only if the parallelogram law

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

holds for every $x, y \in E$.

The equality

$$2(\|A\|_p^p + \|B\|_p^p) = \|A - B\|_p^p + \|A + B\|_p^p$$

holds for $p \neq 2$ if and only if $A^*B = AB^* = 0$, or equivalently the ranges of A and B are orthogonal.

On the other hand, Mc Carthy [13] obtained for $1 < p \le 2$ the following inequality

$$\|A - B\|_p^q + \|A + B\|_p^q \le 2(\|A\|_p^p + \|B\|_p^p)^{q/p}, \tag{1.3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. For $p \ge 2$, this inequality is reversed. Many mathematicians have obtained different generalizations of (1.1) and (1.3) to n-tuples of operators by employing various techniques such as convexity and concavity of certain functions, complex interpolation method, etc.; see [3,4,6,8].

Recently, Audenaert and Kittaneh in [1] have presented a number of conjectures and open problems in the theory of matrix and operator inequalities. More precisely, in Section 8.1 entitled "Clarkson inequalities for several operators" the authors, motivated by the inequalities given by Corollary 2.2 in [6], gave the following conjecture as a new natural generalization of (1.3) to n-tuples of operators.

Audenaert–Kittaneh's Conjecture. Let $A_1, \cdots, A_n \in \mathbb{B}_p(\mathscr{H})$.

(1) For $2 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$n \left(\sum_{j=1}^{n} \|A_j\|_p^p \right)^{q/p} \le \left\| \sum_{j=1}^{n} A_j \right\|_p^q + \sum_{1 \le j < k \le n} \|A_j - A_k\|_p^q. \tag{1.4}$$

(2) For $0 < p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\left\| \sum_{j=1}^{n} A_j \right\|_p^q + \sum_{1 \le j < k \le n} \|A_j - A_k\|_p^q \le n \left(\sum_{j=1}^{n} \|A_j\|_p^p \right)^{q/p}. \tag{1.5}$$

In this paper, we obtain operator inequalities for the p-Schatten class which are a refinement of the identities in [7]; see also [11]. In addition, we obtain an approach to the inequalities conjectured by Audenaert and Kittaneh for the p-Schatten class and in particular we prove that Audenaert–Kittaneh's Conjecture holds at least for the Hilbert–Schmidt norm.

2. Refinements of some p-Schatten inequalities

In this section we present inequalities that can be consider as generalizations of the Clarkson–McCarthy inequalities to multiple arguments, and in particular we work with operators that satisfies an orthogonality condition. More explicitly, we consider $A_i, B_i \in \mathbb{B}_p(\mathscr{H})$ such that $\sum_i A_i$ and $\sum_i B_i$ are orthogonal.

We begin with some lemmas that we use along the paper. The proof of the first one is straightforward and we omit it.

Lemma 2.1. *Let $A_1, \cdots, A_n, B_1, \cdots, B_n \in \mathbb{B}(\mathscr{H})$. If $\sum_{i,j=1}^{n} A_i^* B_j = 0$, then*

$$\sum_{i,j=1}^{n} |A_i \pm B_j|^2 = \sum_{i,j=1}^{n} |A_i|^2 + |B_j|^2 \pm \sum_{i,j=1}^{n} A_i^* B_j + B_j^* A_i = \sum_{i,j=1}^{n} |A_i|^2 + |B_j|^2.$$

The second lemma is rather technical.

Lemma 2.2. *If $A_1, \cdots, A_n \in \mathbb{B}_p(\mathscr{H})$ for some $p > 0$ and $A_1, \cdots, A_n$ are positive, then for $0 < p \le 1$;*

$$n^{p-1}\sum_{i=1}^{n}\|A_i\|_p^p \leq \left(\sum_{i=1}^{n}\|A_i\|_p\right)^p \leq \left\|\sum_{i=1}^{n}A_i\right\|_p^p \leq \sum_{i=1}^{n}\|A_i\|_p^p \tag{2.1}$$

and for $1 \leq p < \infty$ the inequalities are reversed.

Basically, inequalities (2.1) and its reverse inequality were proved in [13] and [2], respectively. They are a refinement of Lemma 2.1 in [7]. These refinements follow from the well-known fact that $\mathcal{M}_s(\bar{x}) \leq \mathcal{M}_{s'}(\bar{x})$ for $0 < s < s'$, where $\mathcal{M}_s(\bar{x}) = (\frac{1}{n}\sum_{i=1}^{n} x_i^s)^{1/s}$ if $\bar{x} = (x_1, \cdots, x_n)$ is an n-tuple of non-negative numbers. A commutative version of the previous lemma for scalars is the following: if $\bar{x} = (x_1, \cdots, x_n)$ is an n-tuple of non-negative numbers, then

$$n^{p-1}\sum_{i=1}^{n}x_i^p \leq \left(\sum_{i=1}^{n}x_i\right)^p \leq \sum_{i=1}^{n}x_i^p \tag{2.2}$$

for $0 < p \leq 1$, and

$$\sum_{i=1}^{n}x_i^p \leq \left(\sum_{i=1}^{n}x_i\right)^p \leq n^{p-1}\sum_{i=1}^{n}x_i^p \tag{2.3}$$

for $1 \leq p < \infty$.

Theorem 2.3. *Let $A_1, \cdots, A_n, B_1, \cdots, B_n \in \mathbb{B}_p(\mathscr{H})$ such that $\sum_{i,j=1}^{n} A_i^* B_j = 0$. Then for $0 < p \leq 2$, $p \leq \lambda$ and $0 < \mu \leq 2$,*

$$\begin{aligned} 2^{\frac{1}{2}-\frac{1}{\mu}} n^{1-\frac{1}{\mu}} \left(\sum_{i=1}^{n}\|A_i\|_p^\mu + \sum_{i=1}^{n}\|B_i\|_p^\mu\right)^{\frac{1}{\mu}} &\leq n^{\frac{1}{2}}\left(\sum_{i=1}^{n}\|A_i\|_p^2 + \sum_{i=1}^{n}\|B_i\|_p^2\right)^{\frac{1}{2}} \\ &\leq n^{2(\frac{1}{p}-\frac{1}{\lambda})}\left(\sum_{i,j=1}^{n}\|A_i \pm B_j\|_p^\lambda\right)^{\frac{1}{\lambda}}. \end{aligned}$$

For $2 \leq p$, $0 < \lambda \leq p$ and $2 \leq \mu$, the inequalities are reversed.

Proof. Let $0 < p \leq 2$, $p \leq \lambda$ and $0 < \mu \leq 2$. It follows from $\mathcal{M}_p(\bar{x}) \leq \mathcal{M}_\lambda(\bar{x})$ that

$$\begin{aligned} n^{2(\frac{1}{p}-\frac{1}{\lambda})}\left(\sum_{i,j=1}^{n}\|A_i \pm B_j\|_p^\lambda\right)^{\frac{1}{\lambda}} &= n^{\frac{2}{p}}\left(\frac{1}{n^2}\sum_{i,j=1}^{n}\|\,|A_i \pm B_j|\,\|_p^\lambda\right)^{\frac{1}{\lambda}} \\ &\geq \left(\sum_{i,j=1}^{n}\|\,|A_i \pm B_j|\,\|_p^p\right)^{\frac{1}{p}}. \end{aligned}$$

Applying the well-known fact that $\|T\|_p^2 = \| \, |T|^2 \, \|_{p/2}$ for any $T \in \mathbb{B}_p(\mathscr{H})$ with $p > 0$ and Lemmas 2.1 and 2.2, we get

$$\begin{aligned}\left(\sum_{i,j=1}^{n} \| \, |A_i \pm B_j| \, \|_p^p\right)^{\frac{1}{p}} &= \left(\sum_{i,j=1}^{n} \| \, |A_i \pm B_j|^2 \, \|_{p/2}^{p/2}\right)^{\frac{1}{p}} \geq \left(\left\|\sum_{i,j=1}^{n} |A_i \pm B_j|^2\right\|_{p/2}^{p/2}\right)^{\frac{1}{p}} \\ &= \left\|\sum_{i,j=1}^{n} |A_i \pm B_j|^2\right\|_{p/2}^{1/2} = \left\|\sum_{i,j=1}^{n} |A_i|^2 + |B_j|^2\right\|_{p/2}^{1/2} \\ &= n^{\frac{1}{2}} \left\|\sum_{i=1}^{n} |A_i|^2 + \sum_{i=1}^{n} |B_i|^2\right\|_{p/2}^{1/2}. \qquad (2.4)\end{aligned}$$

In the following inequalities we use that if $T_1, \dots, T_n$ are positive operators in $\mathbb{B}_p(\mathscr{H})$ then

$$\left\|\sum_{i=1}^{n} T_i\right\|_p \geq \sum_{i=1}^{n} \|T_i\|_p \qquad (2.5)$$

for $0 < p < 1$. This result had been showed by Bhatia and Kittaneh in Lemma 1 and in formula (7) of [2]. Using again Lemma 2.2 and the concavity of the function $f(x) = x^\alpha$ on $[0, +\infty)$ for $0 < \alpha \leq 1$, we obtain

$$\begin{aligned} n^{\frac{1}{2}} \left\|\sum_{i=1}^{n} |A_i|^2 + \sum_{i=1}^{n} |B_i|^2\right\|_{p/2}^{1/2} &= n^{\frac{1}{2}} \left(\left\|\sum_{i=1}^{n} |A_i|^2 + \sum_{i=1}^{n} |B_i|^2\right\|_{p/2}^{\frac{\mu}{2}}\right)^{\frac{1}{\mu}} \\ &\geq n^{\frac{1}{2}} \left(\left(\sum_{i=1}^{n} \| \, |A_i|^2 \|_{p/2} + \sum_{i=1}^{n} \| \, |B_i|^2 \, \|_{p/2}\right)^{\frac{\mu}{2}}\right)^{\frac{1}{\mu}} \\ &\geq n^{\frac{1}{2}} \left((2n)^{\frac{\mu}{2}-1} \left(\sum_{i=1}^{n} \| \, |A_i|^2 \|_{p/2}^{\frac{\mu}{2}} + \sum_{i=1}^{n} \| \, |B_i|^2 \, \|_{p/2}^{\frac{\mu}{2}}\right)\right)^{\frac{1}{\mu}} \\ &= n^{\frac{1}{2}} (2n)^{\frac{1}{2}-\frac{1}{\mu}} \left(\sum_{i=1}^{n} \| \, |A_i|^2 \|_{p/2}^{\frac{\mu}{2}} + \sum_{i=1}^{n} \| \, |B_i|^2 \, \|_{p/2}^{\frac{\mu}{2}}\right)^{\frac{1}{\mu}} \\ &= 2^{\frac{1}{2}-\frac{1}{\mu}} n^{1-\frac{1}{\mu}} \left(\sum_{i=1}^{n} \|A_i\|_p^\mu + \sum_{i=1}^{n} \|B_i\|_p^\mu\right)^{\frac{1}{\mu}}. \qquad \square \qquad (2.6)\end{aligned}$$

Taking $\mu = \lambda = p$ in Theorem 2.3, we obtain the following inequalities which are refinement of several results obtained in [7].

Corollary 2.4. *Let $A_1, \cdots, A_n, B_1, \cdots, B_n \in \mathbb{B}_p(\mathscr{H})$ such that $\sum_{i,j=1}^{n} A_i^* B_j = 0$. Then for $0 < p \leq 2$*

$$2^{\frac{p}{2}-1} n^{p-1} \left(\sum_{i=1}^{n} \|A_i\|_p^p + \sum_{i=1}^{n} \|B_i\|_p^p \right) \leq n^{\frac{p}{2}} \left(\sum_{i=1}^{n} \|A_i\|_p^2 + \sum_{i=1}^{n} \|B_i\|_p^2 \right)^{\frac{p}{2}}$$
$$\leq \sum_{i,j=1}^{n} \|A_i \pm B_j\|_p^p.$$

For $2 \leq p < \infty$ the inequality are reversed.

Corollary 2.5. *Let $A_1, \cdots, A_n \in \mathbb{B}_p(\mathscr{H})$ such that $\sum_{i=1}^{n} A_i = 0$. Then for $0 < p \leq 2$,*

$$2^{\frac{p}{2}} n^{p-1} \sum_{i=1}^{n} \|A_i\|_p^p \leq (2n)^{\frac{p}{2}} \left(\sum_{i=1}^{n} \|A_i\|_p^2 \right)^{\frac{p}{2}} \leq \sum_{i,j=1}^{n} \|A_i \pm A_j\|_p^p. \tag{2.7}$$

For $2 \leq p < \infty$, the inequalities are reversed.

Proof. The hypothesis $\sum_{i=1}^{n} A_i = 0$ implies that $\sum_{i,j=1}^{n} A_i^* A_j = 0$. The statement is therefore a consequence of Corollary 2.4. □

Our second result is a natural generalization of Theorem 2.5 in [7].

Theorem 2.6. *Let $A_1, \cdots, A_n, B_1, \cdots, B_n \in \mathbb{B}_p(\mathscr{H})$ such that $\sum_{i,j=1}^{n} A_i^* B_j = 0$. Then for $0 < p \leq 2$, $p \leq \lambda$ and $0 < \mu \leq 2$,*

$$n \left(\frac{1}{n^2} \sum_{i,j=1}^{n} \|A_i \pm B_j\|_p^\mu \right)^{\frac{1}{\mu}} \leq n^{\frac{1}{2}+\frac{1}{p}-\frac{1}{\lambda}} \left(\sum_{i=1}^{n} \left\| \left(|A_i|^2 + |B_i|^2\right)^{1/2} \right\|_p^\lambda \right)^{\frac{1}{\lambda}}.$$

For $2 \leq p$, $0 < \lambda \leq p$ and $2 \leq \mu$, the inequality is reversed.

Proof. We suppose that $0 < p \leq 2$, $p \leq \lambda$ and $0 < \mu \leq 2$. Then

$$n \left(\frac{1}{n^2} \sum_{i,j=1}^{n} \|A_i \pm B_j\|_p^\mu \right)^{\frac{1}{\mu}} = n \left(\frac{1}{n^2} \sum_{i,j=1}^{n} (\|A_i \pm B_j\|_p^2)^{\mu/2} \right)^{\frac{1}{\mu}}$$
$$= n \left(\frac{1}{n^2} \sum_{i,j=1}^{n} \||A_i \pm B_j|^2\|_{p/2}^{\mu/2} \right)^{\frac{1}{\mu}}$$
$$\leq n \left(\frac{1}{n^2} n^{2(1-\mu/2)} \left(\sum_{i,j=1}^{n} \||A_i \pm B_j|^2\|_{p/2} \right)^{\mu/2} \right)^{\frac{1}{\mu}}$$

$$= \left(\sum_{i,j=1}^{n} \||A_i \pm B_j|^2\|_{p/2} \right)^{\frac{1}{2}}$$

$$\leq \left[n \left(\sum_{i=1} \|(|A_i|^2 + |B_i|^2)^{\frac{1}{2}}\|_p^p \right)^{2/p} \right]^{\frac{1}{2}}$$

$$= \left[n \left(\sum_{i=1} (\|(|A_i|^2 + |B_i|^2)^{\frac{1}{2}}\|_p^{\lambda})^{p/\lambda} \right)^{2/p} \right]^{\frac{1}{2}}$$

$$\leq \left[n \left(n^{1-\frac{p}{\lambda}} \left(\sum_{i=1} \|(|A_i|^2 + |B_i|^2)^{\frac{1}{2}}\|_p^{\lambda} \right)^{p/\lambda} \right)^{2/p} \right]^{\frac{1}{2}}$$

$$= n^{\frac{1}{2}+\frac{1}{p}-\frac{1}{\lambda}} \left(\sum_{i=1}^{n} \left\| \left(|A_i|^2 + |B_i|^2\right)^{1/2} \right\|_p^{\lambda} \right)^{\frac{1}{\lambda}}. \qquad \square$$

3. A conjecture of Audenaert and Kittaneh

In [10] an operator extension of Bohr's inequality is obtained. In particular, it follows from Corollary 2.3 the following statement

Proposition 3.1. *Let $T_1, \cdots, T_n, S_1, \cdots, S_n \in \mathbb{B}(\mathscr{H})$. Then*

$$\sum_{1 \leq j < k \leq n} |T_j - T_k|^2 + \sum_{1 \leq j < k \leq n} |S_j - S_k|^2 = \sum_{j,k=1}^{n} |T_j - S_k|^2 - \left| \sum_{j=1}^{n} T_j - S_j \right|^2. \tag{3.1}$$

Utilizing the previous proposition with $S_k = 0$ for $k = 1, \cdots, n$, we get

$$\sum_{j=1}^{n} |T_j|^2 = \frac{1}{n} \left| \sum_{j=1}^{n} T_j \right|^2 + \frac{1}{n} \sum_{1 \leq j < k \leq n} |T_j - T_k|^2. \tag{3.2}$$

Now we obtain an Audenaert–Kittaneh's Conjecture version with a weaker prefactor $n^{q/2}$ instead of n.

Theorem 3.2. *Let $A_1, \cdots, A_n \in \mathbb{B}_p(\mathscr{H})$. For $2 \leq p < \infty$; $\frac{1}{p} + \frac{1}{q} = 1$,*

$$n^{q/2} \left(\sum_{j=1}^{n} \|A_j\|_p^p \right)^{q/p} \leq \left\| \sum_{j=1}^{n} A_j \right\|_p^q + \sum_{1 \leq j < k \leq n} \|A_j - A_k\|_p^q. \tag{3.3}$$

For $1 < p \leq 2$; $\frac{1}{p} + \frac{1}{q} = 1$, the inequality is reversed.

Proof. We only prove the case when $2 \le p < \infty$; and $\frac{1}{p} + \frac{1}{q} = 1$. The other case can be proved by a similar argument. It follows from Lemma 2.2 that

$$\begin{aligned} n^{q/2}\left(\sum_{j=1}^{n}\|A_j\|_p^p\right)^{q/p} &= \left(n^{p/2}\sum_{j=1}^{n}\|\,|A_j|^2\|_{p/2}^{p/2}\right)^{q/p} \le \left(n^{p/2}\left\|\sum_{j=1}^{n}|A_j|^2\right\|_{p/2}^{p/2}\right)^{q/p} \\ &= \left\|n\sum_{j=1}^{n}|A_j|^2\right\|_{p/2}^{q/2} = \left\|\left|\sum_{j=1}^{n}A_j\right|^2 + \sum_{1\le j<k\le n}|A_j - A_k|^2\right\|_{p/2}^{q/2} \\ &\le \left(\left\|\left|\sum_{j=1}^{n}A_j\right|^2\right\|_{p/2} + \sum_{1\le j<k\le n}\left\|\,|A_j - A_k|^2\right\|_{p/2}\right)^{q/2} \\ &\le \left\|\left|\sum_{j=1}^{n}A_j\right|^2\right\|_{p/2}^{q/2} + \sum_{1\le j<k\le n}\left\|\,|A_j - A_k|^2\right\|_{p/2}^{q/2} \\ &= \left\|\sum_{j=1}^{n}A_j\right\|_p^q + \sum_{1\le j<k\le n}\|A_j - A_k\|_p^q, \end{aligned}$$

which yields the desired inequality. □

Remark 3.3. Note that if $p = q = 2$ then by (3.3) and its reverse inequality, we get

$$n\left(\sum_{j=1}^{n}\|A_j\|_2^2\right) = \left\|\sum_{j=1}^{n}A_j\right\|_2^2 + \sum_{1\le j<k\le n}\|A_j - A_k\|_2^2, \tag{3.4}$$

(we note that this relation is a simple consequence of (3.2)) and in particular if $\sum_{j=1}^{n}A_j = 0$, we have

$$n\left(\sum_{j=1}^{n}\|A_j\|_2^2\right) = \sum_{1\le j<k\le n}\|A_j - A_k\|_2^2. \tag{3.5}$$

Therefore

$$2n\left(\sum_{j=1}^{n}\|A_j\|_2^2\right) = \sum_{j,k=1}^{n}\|A_j - A_k\|_2^2. \tag{3.6}$$

This equality is related to a result due to Lorch [9] about the characterization of inner product spaces. He proved that a normed space X is an inner product space if and only if for a fixed integer $n \geq 3$ and $x_1, \cdots, x_n \in X$ with $\sum_{j=1}^{n} x_j = 0$ we have $2n\left(\sum_{j=1}^{n} \|x_j\|^2\right) = \sum_{j,k=1}^{n} \|x_j - x_k\|^2$.

Acknowledgements

The authors would like to thank the referee for several valuable suggestions and comments. M.S. Moslehian (the corresponding author) was supported by a grant from Ferdowsi University of Mashhad (No. MP94328MOS).

References

[1] K.M.R. Audenaert, F. Kittaneh, Problems and conjectures in matrix and operator inequalities, arXiv:1201.5232v3.
[2] R. Bhatia, F. Kittaneh, Cartesian decompositions and Schatten norms, Linear Algebra Appl. 318 (1–3) (2000) 109–116.
[3] R. Bhatia, F. Kittaneh, Clarkson inequalities with several operators, Bull. Lond. Math. Soc. 36 (6) (2004) 820–832.
[4] C. Conde, Clarkson–McCarthy interpolated inequalities in Finsler norms, JIPAM. J. Inequal. Pure Appl. Math. 10 (1) (2009), article 4, 10 pp.
[5] I. Gohberg, M. Krein, Introduction to the Theory of Linear Nonselfadjoint Operators, vol. 18, American Mathematical Society, Providence, RI, 1969.
[6] O. Hirzallah, F. Kittaneh, Non-commutative Clarkson inequalities for n-tuples of operators, Integral Equations Operator Theory 60 (3) (2008) 369–379.
[7] O. Hirzallah, F. Kittaneh, M.S. Moslehian, Schatten p-norm inequalities related to a characterization of inner product spaces, Math. Inequal. Appl. 13 (2) (2010) 235–241.
[8] E. Kissin, On Clarkson–McCarthy inequalities for n-tuples of operators, Proc. Amer. Math. Soc. 135 (8) (2007) 2483–2495.
[9] E. Lorch, On certain implications which characterize Hilbert space, Ann. of Math. (2) 49 (1948) 523–532.
[10] M.S. Moslehian, An operator extension of the parallelogram law and related norm inequalities, Math. Inequal. Appl. 14 (3) (2011) 717–725.
[11] M.S. Moslehian, M. Tominaga, K.-S. Saito, Schatten p-norm inequalities related to an extended operator parallelogram law, Linear Algebra Appl. 435 (4) (2011) 823–829.
[12] P. Jordan, J. von Neumann, On inner products in linear metric spaces, Ann. of Math. (2) 36 (3) (1935) 719–723.
[13] C. Mc Carthy, c_p, Israel J. Math. 5 (1967) 249–271.