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The non-pure version of the simplex and the boundary of the simplex

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ABSTRACT

We introduce the non-pure versions of simplicial balls and spheres with minimum number of vertices. These are a special type of non-homogeneous balls and spheres (NH-balls and NH-spheres) satisfying a minimality condition on the number of facets. The main result is that *minimal* NH-balls and NH-spheres are precisely the simplicial complexes whose iterated Alexander duals converge respectively to a simplex or the boundary of a simplex.

1. Introduction

A simplicial complex K of dimension d is *vertex-minimal* if it is a d-simplex or it has $d+2$ vertices. It is not hard to see that a vertex-minimal homogeneous (or pure) complex of dimension d is either an elementary starring $(\tau, a)\Delta^d$ of a d-simplex or the boundary $\partial\Delta^{d+1}$ of a $(d+1)$-simplex. On the other hand, a general non-pure complex with minimum number of vertices has no precise characterization. However, since vertex-minimal pure complexes are either balls or spheres, it is natural to ask whether there is a non-pure analogue to these polyhedra within the theory of non-homogeneous balls and spheres. In [5] G. Minian and the author introduced NH-manifolds, a generalization of the concept of manifold to the non-pure setting (somewhat similar to Björner and Wachs's extension of the shellability definition to non-pure complexes [3]). In this theory, NH-balls and NH-spheres are the non-pure versions of combinatorial balls and spheres.

The purpose of this article is to study *minimal* NH-balls and NH-spheres, which are respectively the non-pure counterpart of vertex-minimal balls and spheres. Note that $\partial\Delta^{d+1}$ is not only the d-sphere with minimum number of vertices but also the one with minimum number of facets. For non-pure spheres, this last property is strictly stronger than vertex-minimality and it is convenient to define minimal NH-spheres as the ones with minimum number of facets. With this definition, minimal NH-spheres with the homotopy type of a k-sphere are precisely the non-pure spheres whose nerve is $\partial\Delta^{k+1}$, a property that also characterizes the boundary of simplices. On the other hand, an NH-ball B is minimal if it is part of a decomposition of a minimal NH-sphere, i.e. if there exists a combinatorial ball L with $B\cap L=\partial L$ such that $B+L$ is a minimal NH-sphere. This definition is consistent with the notion of vertex-minimal simplicial ball (see Lemma 4.1 below).

Surprisingly, minimal NH-balls and NH-spheres can be characterized by a property involving Alexander duals. Denote by K^* the Alexander dual of a complex K relative to the vertices of K. Set inductively $K^{*(0)}=K$ and $K^{*(m)}=(K^{*(m-1)})^*$. Thus, in each step $K^{*(i)}$ is computed relatively to its own vertices, i.e. as a subcomplex of the boundary of the simplex of minimum dimension containing it. We call $(K^{*(m)})_{m\in\mathbb{N}_0}$ the *sequence of iterated Alexander duals* of K. The main result of the article is the following.

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Theorem 1.1. *Let K be a finite simplicial complex.*

(*i*) *There is an $m \in \mathbb{N}_0$ such that $K^{*(m)}$ is the boundary of a simplex if and only if K is a minimal NH-sphere.*
(*ii*) *There is an $m \in \mathbb{N}_0$ such that $K^{*(m)}$ is a simplex if and only if K is a minimal NH-ball.*

In any case, the number of iterations needed to reach the simplex or the boundary of the simplex is bounded above by the number of vertices of K.

Note that $K^* = \Delta^d$ if and only if K is a vertex-minimal d-ball which is not a simplex, so (*ii*) describes precisely all complexes *converging* to vertex-minimal balls. Theorem 1.1 characterizes the classes of Δ^d and $\partial\Delta^d$ in the equivalence relation generated by $K \sim K^*$.

2. Preliminaries

2.1. Notation and definitions

All simplicial complexes that we deal with are assumed to be finite. Given a set of vertices V, $|V|$ will denote its cardinality and $\Delta(V)$ the simplex spanned by its vertices. $\Delta^d := \Delta(\{0, \dots, d\})$ will denote a generic d-simplex and $\partial\Delta^d$ its boundary. The set of vertices of a complex K will be denoted V_K and we set $\Delta_K := \Delta(V_K)$. A *facet* of a complex K is a simplex which is not a proper face of any other simplex of K. We denote by $\mathsf{f}(K)$ the number of facets in K. A *ridge* is a maximal proper face of a facet. A complex is *pure* or *homogeneous* if all its facets have the same dimension.

We denote by $\sigma * \tau$ the join of the faces $\sigma, \tau \in K$ (if $\sigma \cap \tau = \emptyset$) and by $K * L$ the join of the complexes K and L (if $V_K \cap V_L = \emptyset$). By convention, if $\emptyset$ is the empty simplex and $\{\emptyset\}$ the complex containing only the empty simplex then $K * \{\emptyset\} = K$ and $K * \emptyset = \emptyset$. Note that $\partial\Delta^0 = \{\emptyset\}$. For $\sigma \in K$, $lk(\sigma, K) = \{\tau \in K : \ \tau \cap \sigma = \emptyset, \ \tau * \sigma \in K\}$ denotes its *link* and $st(\sigma, K) = \sigma * lk(\sigma, K)$ its *star*. The union of two complexes K, L will be denoted by $K + L$. A subcomplex $L \subseteq K$ is said to be *top generated* if every facet of L is also a facet of K.

The notation $K \searrow L$ will mean that K (simplicially) collapses to L. A complex is *collapsible* if it collapses to a single vertex and *PL-collapsible* if it has a subdivision which is collapsible. The *simplicial nerve* $\mathcal{N}(K)$ of K is the complex whose vertices are the facets of K and whose simplices are the finite subsets of facets of K with non-empty intersection.

Two complexes are *PL-isomorphic* if they have a common subdivision. A *combinatorial d-ball* is a complex *PL*-isomorphic to Δ^d. A *combinatorial d-sphere* is a complex *PL*-isomorphic to $\partial\Delta^{d+1}$. By convention, $\partial\Delta^0 = \{\emptyset\}$ is a sphere of dimension -1. A *combinatorial d-manifold* is a complex M such that $lk(v, M)$ is a combinatorial $(d-1)$-ball or $(d-1)$-sphere for every $v \in V_M$. A $(d-1)$-simplex in a combinatorial d-manifold M is a face of at most two d-simplices of M and the boundary ∂M is the complex generated by the $(d-1)$-simplices which are faces of exactly one d-simplex. Combinatorial d-balls and d-spheres are combinatorial d-manifolds. The boundary of a combinatorial d-ball is a combinatorial $(d-1)$-sphere.

2.2. Non-homogeneous balls and spheres

In order to make the presentation self-contained, we recall first the definition and some basic properties of non-homogeneous balls and spheres. For a comprehensive exposition of the subject, the reader is referred to [5] (see also [6, §2.2] for a brief summary).

NH-balls and NH-spheres are special types of NH-manifolds, which are the non-pure versions of combinatorial manifolds. NH-manifolds have a local structure consisting of regularly-assembled pieces of Euclidean spaces of different dimensions. In Fig. 1 we show some examples of NH-manifolds and their underlying spaces. NH-manifolds, NH-balls and NH-spheres are defined as follows.

Definition. An *NH-manifold* (resp. *NH-ball*, *NH-sphere*) of dimension 0 is a combinatorial manifold (resp. ball, sphere) of dimension 0. An NH-sphere of dimension -1 is, by convention, the complex $\{\emptyset\}$. For $d \geq 1$, we define by induction:

- An *NH-manifold* of dimension d is a complex M of dimension d such that $lk(v, M)$ is an NH-ball or an NH-sphere (possibly of dimension -1) for all $v \in V_M$.
- An *NH-ball* of dimension d is a PL-collapsible NH-manifold of dimension d.
- An *NH-sphere* of dimension d and *homotopy dimension* k is an NH-manifold S of dimension d such that there exist a top generated NH-ball B of dimension d and a top generated combinatorial k-ball L such that $B + L = S$ and $B \cap L = \partial L$. We say that $S = B + L$ is a *decomposition* of S and write $\dim_h(S)$ for the homotopy dimension of S.

The definitions of NH-ball and NH-sphere are motivated by the classical theorems of Whitehead [9] and Newman [7] (see e.g. [8, Corollaries 3.28 and 3.13]). Just like for classical combinatorial manifolds, it can be seen that the class of NH-manifolds (resp. NH-balls, NH-spheres) is closed under subdivision and that the link of *every* simplex in an NH-manifold is an NH-ball or an NH-sphere. Also, the homogeneous NH-manifolds (resp. NH-balls, NH-spheres) are

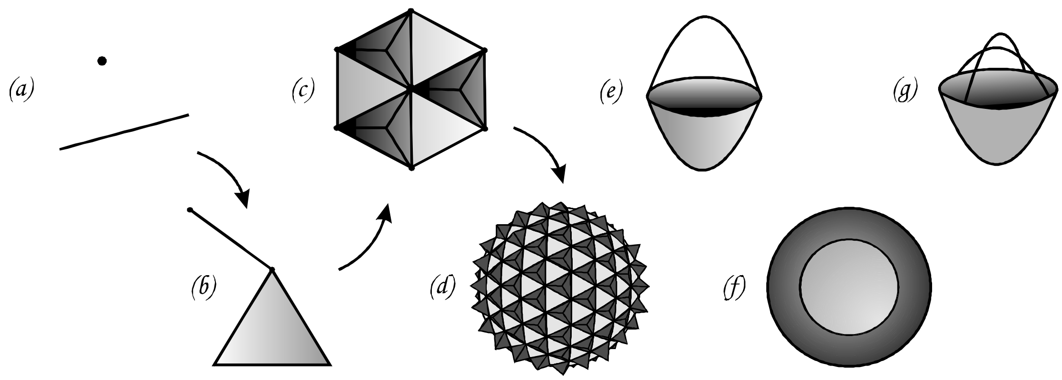

Fig. 1. Examples of NH-manifolds (dark gray areas are 3-dimensional). (a), (d) and (e) are NH-spheres of dimension 1, 3 and 2 and homotopy dimension 0, 2 and 1 respectively. (b) is an NH-ball of dimension 2 and (c), (f) are NH-balls of dimension 3. (g) is an NH-manifold which is neither an NH-ball nor an NH-sphere. The sequence (a)–(d) evidences how NH-manifolds are inductively defined.

precisely the combinatorial manifolds (resp. balls, spheres). Globally, a connected NH-manifold M is (non-pure) *strongly connected*: given two facets $\sigma, \tau \in M$ there is a sequence of facets $\sigma = \eta_1, \dots, \eta_t = \tau$ such that $\eta_i \cap \eta_{i+1}$ is a ridge of η_i or η_{i+1} for every $1 \le i \le t-1$ (see [5, Lemma 3.15]). In particular, NH-balls and NH-spheres of homotopy dimension greater that 0 are strongly connected.

Unlike for classical spheres, non-pure NH-spheres do have boundary simplices; that is, simplices whose links are NH-balls. However, for any decomposition $S = B + L$ of an NH-sphere and any $\sigma \in B \cap L$, $lk(\sigma, S)$ is an NH-sphere with decomposition $lk(\sigma, S) = lk(\sigma, B) + lk(\sigma, L)$ (see [5, Lemma 4.8]). In particular, if $\sigma \in B \cap L$ then $lk(\sigma, B)$ is an NH-ball.

Remark 2.1. Note that the "combinatorial" adjective may be safely removed from the previous remarks since a triangulated manifold all of whose simplices' links are homeomorphic to spheres or balls is a combinatorial manifold (see the proof of [5, Theorem 3.6]). In particular, pure NH-balls are necessarily combinatorial balls since collapsible non-balls cannot occur in the combinatorial setting.

2.3. The Alexander dual

For a finite simplicial complex K and a ground set of vertices $V \supseteq V_K$, the *Alexander dual* of K (relative to V) is the complex

$$K^{*_V} = \{\sigma \subseteq V \mid V \setminus \sigma \notin K\}.$$

The main importance of K^{*_V} lies in the combinatorial formulation of Alexander duality: $H_i(K^{*_V}) \simeq H^{n-i-3}(K)$. Here $n = |V|$ and the homology and cohomology groups are reduced (see e.g. [1,2]). In what follows, we shall write $K^* := K^{*_{V_K}}$ and $K^\tau := K^{*_V}$ if $\tau = V \setminus V_K$. With this convention, $K^\tau = K^*$ if $\tau = \emptyset$. Note that $(\Delta^d)^* = \emptyset$ and $(\partial\Delta^{d+1})^* = \{\emptyset\}$.

The relationship between Alexander duals relative to different ground sets of vertices is given by the following formula (see [6, Lemma 3.2]):

$$K^\tau = \partial\tau * \Delta_K + \tau * K^*. \qquad (*)$$

Here K^* is viewed as a subcomplex of Δ_K. It is easy to see from the definition that $(K^*)^{V_K \setminus V_{K^*}} = K$ and that $(K^\tau)^* = K$ if $K \neq \Delta^d$ (see [6, Lemma 3.2]). The following result characterizes the Alexander dual of vertex-minimal complexes.

Lemma 2.2 *([6, Lemma 3.6]). If $K = \Delta^d + u * lk(u, K)$ with $u \notin \Delta^d$, then $K^* = lk(u, K)^\tau$ where $\tau = V_K \setminus V_{st(u,K)}$.*

It can be shown that K^τ is an NH-ball (resp. NH-sphere) if and only if K^* is an NH-ball (resp. NH-sphere). This actually follows from the next result involving a slightly more general form of formula $(*)$, which we include here for future reference.

Lemma 2.3 *([6, Lemma 5.1]). If $V_K \subseteq V$ and $\eta \neq \emptyset$, then $L := \partial\eta * \Delta(V) + \eta * K$ is an NH-ball (resp. NH-sphere) if and only if K is an NH-ball (resp. NH-sphere).*

3. Minimal NH-spheres

In this section we introduce the non-pure version of $\partial\Delta^d$ and prove part (i) of Theorem 1.1. Recall that $\mathrm{f}(K)$ denotes the number of facets of K. We shall see that for a non-homogeneous sphere S, requesting minimality of $\mathrm{f}(S)$ is strictly stronger than requesting that of V_S. This is the reason why vertex-minimal NH-spheres are not necessarily *minimal* in our sense.

To introduce minimal NH-spheres we note first that any complex K with the homotopy type of a k-sphere has at least $k+2$ facets. This follows from the fact that the simplicial nerve $\mathcal{N}(K)$ is homotopy equivalent to K.

Definition. An NH-sphere S is said to be *minimal* if $\mathsf{f}(S) = \dim_h(S) + 2$.

Note that, equivalently, an NH-sphere S of homotopy dimension k is minimal if and only if $\mathcal{N}(S) = \partial\Delta^{k+1}$.

Remark 3.1. Suppose $S = B + L$ is a decomposition of a minimal NH-sphere of homotopy dimension k and let $v \in V_L$. Then $lk(v, S)$ is an NH-sphere of homotopy dimension $\dim_h(lk(v, S)) = k - 1$ and $lk(v, S) = lk(v, B) + lk(v, L)$ is a valid decomposition (see §2.2). In particular, $\mathsf{f}(lk(v, S)) \geq k + 1$. Also, $\mathsf{f}(lk(v, S)) < k + 3$ since $\mathsf{f}(S) < k + 3$ and $\mathsf{f}(lk(v, S)) \neq k + 2$ since otherwise S is a cone. Therefore, $\mathsf{f}(lk(v, S)) = k + 1 = \dim_h(lk(v, S)) + 2$, which shows that $lk(v, S)$ is also a minimal NH-sphere.

We next prove that minimal NH-spheres are vertex-minimal.

Proposition 3.2. *If S is a d-dimensional minimal NH-sphere then $|V_S| = d + 2$.*

Proof. Let $S = B + L$ be decomposition of S and set $k = \dim_h(S)$. We shall prove that $|V_S| \leq d + 2$ by induction on k. The case $k = 0$ is straightforward, so assume $k \geq 1$. Let $\eta \in B$ be a facet of minimal dimension and let ω denote the intersection of all facets of S different from η. Note that $\omega \neq \emptyset$ since $\mathcal{N}(S) = \partial\Delta^{k+1}$ and let $u \in \omega$ be a vertex. Since $\eta \notin L$ then $\omega \in L$ and hence $u \in L$. By Remark 3.1, $lk(u, S)$ is a minimal NH-sphere of dimension $d' \leq d - 1$ and homotopy dimension $k - 1$. By inductive hypothesis, $|V_{lk(u,S)}| \leq d' + 2 \leq d + 1$. Therefore, $st(u, S)$ is a top generated subcomplex of S with $k + 1$ facets and at most $d + 2$ vertices. By construction, $S = st(u, S) + \eta$. We shall show that $V_\eta \subset V_{st(u,S)}$. Since $B = st(u, B) + \eta$, by strong connectivity there is a ridge $\sigma \in B$ in $st(u, B) \cap \eta$ (see §2.2). By the minimality of η we must have $\eta = w * \sigma$ for some vertex w. Now, $\sigma \in st(u, B) \cap \eta \subset st(u, S) \cap \eta$; but $st(u, S) \cap \eta \neq \sigma$ since, otherwise, $S = st(u, S) + \eta \searrow st(u, S) \searrow u$, contradicting the fact that S has the homotopy type of a sphere. We conclude that $w \in st(u, S)$ since every face of η not contained in σ contains w. Thus, $|V_S| = |V_{st(u,S)} \cup V_\eta| = |V_{st(u,S)}| \leq d + 2$. □

This last proposition shows that, in the non-pure setting, requesting the minimality of $\mathsf{f}(S)$ is strictly more restrictive than requesting that of $|V_S|$. For example, a vertex-minimal NH-sphere can be constructed from *any* NH-sphere S and a vertex $u \notin V_S$ by the formula $\tilde{S} := \Delta_S + u * S$. It is easy to see that if S is not minimal, neither is $\tilde{S}$.

Remark 3.3. By Proposition 3.2, a d-dimensional minimal NH-sphere S may be written $S = \Delta^d + u * lk(u, S)$ for some $u \notin \Delta^d$. Note that for any decomposition $S = B + L$, the vertex u must lie in L (since this last complex is top generated). In particular, $lk(u, S)$ is a minimal NH-sphere by Remark 3.1.

As we mentioned above, the Alexander duals play a key role in characterizing minimal NH-spheres. We now turn to prove Theorem 1.1 (i). We derive first the following corollary of Proposition 3.2.

Corollary 3.4. *If S is a minimal NH-sphere then $|V_{S^*}| < |V_S|$ and $\dim(S^*) < \dim(S)$.*

Proof. $V_{S^*} \subsetneq V_S$ follows from Proposition 3.2 since if $S = \Delta^d + u * lk(u, S)$ then $u \notin S^*$. In particular, this implies that $\dim(S^*) \neq \dim(S)$ since S^* is not a simplex by Alexander duality. □

Theorem 3.5. *Let K be a finite simplicial complex and let τ be a simplex (possibly empty) disjoint from K. Then, K is a minimal NH-sphere if and only if K^τ is a minimal NH-sphere. That is, the class of minimal NH-spheres is closed under taking Alexander dual.*

Proof. Assume first that K is a minimal NH-sphere and set $d = \dim(K)$. We proceed by induction on d. By Proposition 3.2, we can write $K = \Delta^d + u * lk(u, K)$ for some vertex $u \notin \Delta^d$. If $\tau = \emptyset$ then, by Lemma 2.2, $K^* = lk(u, K)^\rho$ for $\rho = V_K \setminus V_{st(u,K)}$. By Remark 3.3, $lk(u, K)$ is a minimal NH-sphere. Therefore, $K^* = lk(u, K)^\rho$ is a minimal NH-sphere by inductive hypothesis. If $\tau \neq \emptyset$, $K^\tau = \partial\tau * \Delta_K + \tau * K^*$ by formula ($*$). In particular, K^τ is an NH-sphere by Lemma 2.3 and the case $\tau = \emptyset$. Now, by Alexander duality,

$$\dim_h(K^\tau) = |V_K \cup V_\tau| - \dim_h(K) - 3 = |V_K| + |V_\tau| - \dim_h(K) - 3 = \dim_h(K^*) + |V_\tau|.$$

On the other hand,

$$\mathsf{f}(K^\tau) = \mathsf{f}(\partial\tau * \Delta_K + \tau * K^*) = \mathsf{f}(\partial\tau) + \mathsf{f}(K^*) = |V_\tau| + \dim_h(K^*) + 2,$$

where the last equality follows from the case $\tau = \emptyset$. This shows that K^τ is minimal.

Assume now that K^τ is a minimal NH-sphere. If $\tau \neq \emptyset$ then $K = (K^\tau)^*$ and if $\tau = \emptyset$ then $K = (K^*)^{V_K \setminus V_{K^*}}$ (see §2.3). In any case, the result follows immediately from the previous implication. □

Proof of Theorem 1.1 (*i*). Suppose first that K is a minimal NH-sphere. By Theorem 3.5, every non-empty complex in the sequence $\{K^{*(m)}\}_{m \in \mathbb{N}_0}$ is a minimal NH-sphere. By Corollary 3.4, $|V_{K^{*(m+1)}}| < |V_{K^{*(m)}}|$ for all m such that $K^{*(m)} \neq \{\emptyset\}$. Therefore, $K^{*(m_0)} = \{\emptyset\}$ for some $m_0 < |V_K|$ and hence $K^{*(m_0-1)} = \partial\Delta^d$ for some $d \geq 1$.

Assume now that $K^{*(m)} = \partial\Delta^d$ for some $m \in \mathbb{N}_0$ and $d \geq 1$. We proceed by induction on m. The case $m = 0$ corresponds to the trivial case $K = \partial\Delta^d$. For $m \geq 1$, the result follows immediately from Theorem 3.5 and the inductive hypothesis. □

4. Minimal *NH*-balls

We now develop the notion of minimal NH-ball. The definition in this case is a little less straightforward than in the case of spheres because there is no piecewise-linear-equivalence argument in the construction of non-pure balls. To motivate the definition of minimal NH-ball, recall that for a non-empty simplex $\tau \in K$ and a vertex $a \notin K$, the *elementary starring* (τ, a) of K is the operation which transforms K in $(\tau, a)K$ by removing $\tau * lk(\tau, K) = st(\tau, K)$ and replacing it with $a * \partial\tau * lk(\tau, K)$. Note that when $\dim(\tau) = 0$ then $(\tau, a)K$ is isomorphic to K.

Lemma 4.1. *Let B be a combinatorial d-ball. The following statements are equivalent.*

(1) $|V_B| \leq d+2$ (i.e. B is vertex-minimal).
(2) B is an elementary starring of Δ^d.
(3) $B \subset \partial\Delta^{d+1}$.
(4) There is a combinatorial d-ball L such that $B + L = \partial\Delta^{d+1}$ and $B \cap L = \partial L$.

Proof. We first prove that (1) implies (2) by induction on d. Since Δ^d is trivially a starring of any of its vertices, we may assume $|V_B| = d+2$ and write $B = \Delta^d + u * lk(u, B)$ for some vertex $u \notin \Delta^d$. Since $lk(u, B)$ is necessarily a vertex-minimal $(d-1)$-combinatorial ball then $lk(u, B) = (\tau, a)\Delta^{d-1}$ by inductive hypothesis. It follows from an easy computation that B is isomorphic to $(u * \tau, a)\Delta^d$.

We next prove that (2) implies (4). We have

$$B = (\tau, a)\Delta^d = a * \partial\tau * lk(\tau, \Delta^d) = a * \partial\tau * \Delta^{d-\dim(\tau)-1} = \partial\tau * \Delta^{d-\dim(\tau)}.$$

Letting $L := \tau * \partial\Delta^{d-\dim(\tau)}$ we get $B + L = \partial\Delta^{d+1}$ and

$$B \cap L = \partial\tau + \partial\Delta^{d-\dim(\tau)} = \partial(\tau * \partial\Delta^{d-\dim(\tau)}) = \partial L.$$

Finally, (4) trivially implies (3) and (1) trivially follows from (3). □

Definition. An NH-ball B is said to be *minimal* if there exists a minimal NH-sphere S that admits a decomposition $S = B + L$.

Note that if B is a minimal NH-ball and $S = B + L$ is a decomposition of a minimal NH-sphere then, by Remark 3.1, $lk(v, B)$ is a minimal NH-ball for every $v \in B \cap L$ (see §2.2). Note also that the intersection of all the facets of B is non-empty since $\mathcal{N}(B) \subsetneq \mathcal{N}(S) = \partial\Delta^{k+1}$. Therefore, $\mathcal{N}(B)$ is a simplex. The converse, however, is easily seen to be false.

The proof of Theorem 1.1 (*ii*) will follow the same lines as its version for NH-spheres.

Proposition 4.2. *If B is a d-dimensional minimal NH-ball then $|V_B| \leq d+2$.*

Proof. This follows immediately from Proposition 3.2 since $\dim(B) = \dim(S)$ for any decomposition $S = B + L$ of an NH-sphere. □

Corollary 4.3. *If B is a minimal NH-ball then $|V_{B^*}| < |V_B|$ and $\dim(B^*) < \dim(B)$.*

Proof. We may assume $B \neq \Delta^d$. $V_{B^*} \subsetneq V_B$ by the same reasoning made in the proof of Corollary 3.4. Also, if $\dim(B) = \dim(B^*)$ then $B^* = \Delta^d$. By formula $(*)$, $B = (B^*)^\rho = \partial\rho * \Delta^d$ where $\rho = V_B \setminus V_{B^*}$, which is a contradiction since $|V_B| = d+2$. □

Remark 4.4. The same construction that we made for minimal NH-spheres shows that vertex-minimal NH-balls need not be minimal. Also, similarly to the case of non-pure spheres, if $B = \Delta^d + u * lk(u, B)$ is a minimal NH-ball which is not a simplex then for any decomposition $S = B + L$ of a minimal NH-sphere we have $u \in L$. In particular, since $lk(u, S) = lk(u, B) + lk(u, L)$ is a valid decomposition of a minimal NH-sphere, then $lk(u, B)$ is a minimal NH-ball (see Remark 3.3).

Theorem 4.5. *Let K be a finite simplicial complex and let τ be a simplex (possibly empty) disjoint from K. Then, K is a minimal NH-ball if and only if K^τ is a minimal NH-ball. That is, the class of minimal NH-balls is closed under taking Alexander dual.*

Proof. Assume first that K is a minimal NH-ball and proceed by induction on $d = \dim(K)$. The case $\tau = \emptyset$ follows the same reasoning as the proof of Theorem 3.5 using the previous remarks. Suppose then $\tau \neq \emptyset$. Since by the previous case K^* is a minimal NH-ball, there exists a decomposition $\tilde{S} = K^* + \tilde{L}$ of a minimal NH-sphere. By Proposition 3.2 and Proposition 4.2, either K^* is a simplex (and $V_{\tilde{S}} \setminus V_{K^*} = \{w\}$ is a single vertex) or $V_{\tilde{S}} = V_{K^*} \subset V_K$. Let $S := K^\tau + \tau * \tilde{L}$, where we identify the vertex w with any vertex in $V_K \setminus V_{K^*}$ if K^* is a simplex. We claim that $S = K^\tau + \tau * \tilde{L}$ is a valid decomposition of a minimal NH-sphere. On one hand, formula $(*)$ and Lemma 2.3 imply that K^τ is an NH-ball and that

$$S = \partial\tau * \Delta_K + \tau * K^* + \tau * \tilde{L} = \partial\tau * \Delta_K + \tau * \tilde{S}$$

is an NH-sphere. Also,

$$\begin{aligned} K^\tau \cap (\tau * \tilde{L}) &= (\partial\tau * \Delta_K + \tau * K^*) \cap (\tau * \tilde{L}) \\ &= \partial\tau * \tilde{L} + \tau * (K^* \cap \tilde{L}) \\ &= \partial\tau * \tilde{L} + \tau * \partial\tilde{L} \\ &= \partial(\tau * \tilde{L}). \end{aligned}$$

This shows that $S = K^\tau + \tau * \tilde{L}$ is valid decomposition of an NH-sphere. On the other hand,

$$\mathsf{f}(S) = \mathsf{f}(\partial\tau) + \mathsf{f}(\tilde{S}) = \dim(\tau) + 1 + \dim(\tilde{L}) + 2 = \dim_h(S) + 2,$$

which proves that S is minimal. This settles the implication.

The other implication is analogous to the corresponding part of the proof of Theorem 3.5. □

Proof of Theorem 1.1 (*ii*). It follows the same reasoning as the proof of Theorem 1.1 (i) (replacing $\{\emptyset\}$ with $\emptyset$). □

If $K^* = \Delta^d$ then, letting $\tau = V_K \setminus V_{\Delta^d} \neq \emptyset$, we have $K = (K^*)^\tau = \partial\tau * \Delta^d = (\tau, v)\Delta^{d+\dim(\tau)}$. This shows that Theorem 1.1 (ii) characterizes all complexes which converge to vertex-minimal balls.

5. Further properties of minimal NH-balls and NH-spheres

In this final section we briefly discuss some characteristic properties of minimal NH-balls and NH-spheres.

Proposition 5.1. *In a minimal NH-ball or NH-sphere, the link of every simplex is a minimal NH-ball or NH-sphere.*

Proof. Let K be a minimal NH-ball or NH-sphere of dimension d and let $\sigma \in K$. We may assume $K \neq \Delta^d$. Since for a non-trivial decomposition $\sigma = w * \eta$ we have $lk(\sigma, S) = lk(w, lk(\eta, S))$, by an inductive argument it suffices to prove the case $\sigma = v \in V_K$. We proceed by induction on d. We may assume $d \geq 1$. Write $K = \Delta^d + u * lk(u, K)$ where, as shown before, $lk(u, K)$ is either a minimal NH-ball or a minimal NH-sphere. Note that this in particular settles the case $v = u$. Suppose then $v \neq u$. If $v \notin lk(u, K)$ then $lk(v, K) = \Delta^{d-1}$. Otherwise, $lk(v, K) = \Delta^{d-1} + u * lk(v, lk(u, K))$. By inductive hypothesis, $lk(v, lk(u, K))$ is a minimal NH-ball or NH-sphere. By Lemma 2.2,

$$lk(v, K)^* = lk(v, lk(u, K))^\rho,$$

and the result follows from Theorem 3.5 and Theorem 4.5. □

For any vertex $v \in K$, the *deletion* $K - v := \{\sigma \in K \mid v \notin \sigma\}$ is again a minimal NH-ball or NH-sphere. This follows from Proposition 5.1, Theorem 3.5, Theorem 4.5 and the fact that $lk(v, K^*) = (K - v)^*$ for any $v \in V_K$ (see [6, Lemma 3.7 (1)]). We can also show that minimal NH-balls are (non-pure) vertex-decomposable as defined by Björner and Wachs [4]. Recall that a complex K is *vertex-decomposable* if

(1) K is a simplex or $K = \{\emptyset\}$, or
(2) there exists a vertex $v \in K$ (called *shedding vertex*) such that
  (a) $K - v$ and $lk(v, K)$ are vertex-decomposable and
  (b) no facet of $lk(v, K)$ is a facet of $K - v$.

Thus, if $B = \Delta^d + u * lk(u, B)$ is a minimal NH-ball which is not a simplex then u is a shedding vertex by Remark 4.4 and an inductive argument on $\dim(B)$. In particular, minimal NH-balls are collapsible (see [4, Theorem 11.3]).

We next make use of Theorem 3.5 and Theorem 4.5 to compute (up to isomorphism) the number of minimal NH-spheres and NH-balls in each dimension.

Proposition 5.2. *Let $0 \leq k \leq d$.*

(1) There are exactly $\binom{d}{k}$ minimal NH-spheres of dimension d and homotopy dimension k. In particular, there are exactly 2^d minimal NH-spheres of dimension d.
(2) There are exactly 2^d minimal NH-balls of dimension d.

Proof. We first prove (1). An NH-sphere with $d = k$ is homogeneous by [6, Proposition 2.7], in which case the result is obvious. Assume then $0 \leq k \leq d-1$ and proceed by induction on d. Let $\mathcal{S}_{d,k}$ denote the set of minimal NH-spheres of dimension d and homotopy dimension k. If $S \in \mathcal{S}_{d,k}$ it follows from Theorem 3.5, Corollary 3.4 and Alexander duality that S^* is a minimal NH-sphere with $\dim(S^*) < d$ and $\dim_h(S^*) = d-k-1$. Therefore, there is a well defined application

$$\mathcal{S}_{d,k} \xrightarrow{f} \bigcup_{i=d-k-1}^{d-1} \mathcal{S}_{i,d-k-1}$$

sending S to S^*. We claim that f is a bijection. To prove injectivity, suppose $S_1, S_2 \in \mathcal{S}_{d,k}$ are such that $S_1^* = S_2^*$. Let $\rho_i = V_{S_i} \setminus V_{S_i^*}$ $(i = 1, 2)$. Since $|V_{S_1}| = d+2 = |V_{S_2}|$ then $\dim(\rho_1) = \dim(\rho_2)$ and, hence, $S_1 = (S_1^*)^{\rho_1} = (S_2^*)^{\rho_2} = S_2$. To prove surjectivity, let $\tilde{S} \in \mathcal{S}_{j,d-k-1}$ with $d-k-1 \leq j \leq d-1$. Taking $\tau = \Delta^{d-j-1}$ we have $\tilde{S}^\tau \in \mathcal{S}_{d,k}$ and $f(\tilde{S}^\tau) = \tilde{S}$ (see §2.3). Finally, using the inductive hypothesis,

$$|\mathcal{S}_{d,k}| = \sum_{i=d-k-1}^{d-1} |\mathcal{S}_{i,d-k-1}| = \sum_{i=d-k-1}^{d-1} \binom{i}{d-k-1} = \binom{d}{k}.$$

For (2), let $\mathcal{B}_d$ denote the set of minimal NH-balls of dimension d and proceed again by induction on d. The very same reasoning as above gives a well defined bijection

$$\mathcal{B}_d \setminus \{\Delta^d\} \xrightarrow{f} \bigcup_{i=0}^{d-1} \mathcal{B}_i.$$

Therefore, using the inductive hypothesis,

$$|\mathcal{B}_d \setminus \{\Delta^d\}| = \sum_{i=0}^{d-1} |\mathcal{B}_i| = \sum_{i=0}^{d-1} 2^i = 2^d - 1. \qquad \square$$

Finally, we give a direct combinatorial description of minimal NH-balls and NH-spheres. This description (and its proof) was suggested by an anonymous referee. We are very grateful to him/her for this contribution.

Let $V = \{v_1, \ldots, v_t\} \neq \emptyset$ and W be disjoint sets of vertices. Given a collection $\mathcal{H} = \{H_1, \ldots, H_t\}$ of subsets of W, we let $K(V, W, \mathcal{H}) \subset \Delta(V \cup W)$ be the simplicial complex whose facets are the simplices $\eta_i := (V \setminus \{v_i\}) \cup H_i$ for $1 \leq i \leq t$. Note that

$$V_{K(V,W,\mathcal{H})} = \begin{cases} V \cup W & t \geq 2 \\ H_t & t = 1. \end{cases}$$

Proposition 5.3. *Let K be a simplicial complex. Then*

(1) K is a minimal NH-sphere of dimension d and homotopy dimension k if and only if K is isomorphic to $K(V, W, \mathcal{H})$ for vertex sets $V = \{v_1, \ldots, v_{k+2}\}$ and $W = \{w_1, \ldots, w_{d-k}\}$ and a collection $\mathcal{H} = \{H_1, \ldots, H_{k+2}\}$ satisfying $\emptyset = H_1 \subseteq H_2 \subseteq \cdots \subseteq H_{k+2} = W$.
(2) K is a minimal NH-ball of dimension d if and only if K is isomorphic to $K(V, W, \mathcal{H})$ for vertex sets $V = \{v_1, \ldots, v_t\}$ $(t \leq d+1)$ and $W = \{w_1, \ldots, w_{d+2-t}\}$ and a collection $\mathcal{H} = \{H_1, \ldots, H_t\}$ satisfying $\emptyset \neq H_1 \subseteq H_2 \subseteq \cdots \subseteq H_t = W$.

Proof. We deal with (1) first. Let K be a minimal NH-sphere of dimension d and homotopy dimension k and let $\eta_1, \ldots, \eta_{k+2}$ be the facets of K. Since $\mathcal{N}(K) = \partial\Delta^{k+1}$ then, for all $1 \leq i \leq k+2$, there is a vertex $v_i \in \bigcap_{j \neq i} \eta_j$ (and then $v_i \notin \eta_i$). Set $V := \{v_1, \ldots, v_{k+2}\}$ and let $W := V_K \setminus V$. We further set $H_i := V_{\eta_i} \cap W$. By relabeling the η_i's we may assume that $|H_1| \leq |H_2| \leq \cdots \leq |H_{k+2}|$. Note that $\eta_i = (V \setminus \{v_i\}) \cup H_i$ and that $|W| = d-k$ by Proposition 3.2. It remains to show that $\emptyset = H_1 \subseteq H_2 \subseteq \cdots \subseteq H_{k+2} = W$. On one hand, $H_1 = \emptyset$ since K has k-dimensional facets and $H_{k+2} = W$ since $\dim(K) = d$. On the other hand, if $H_i \nsubseteq H_j$ for some $i < j$, then, given that $|H_i| \leq |H_j|$, there are vertices $w_i \in H_i \setminus H_j$ and $w_j \in H_j \setminus H_i$. Let $\rho = V \setminus \{v_i, v_j\}$. Note that since the only facets of K containing ρ are η_i and η_j then $lk(\rho, K) = (v_j * \Delta(H_i)) + (v_i * \Delta(H_j))$. Consider $L := lk(H_i \cap H_j, lk(\rho, K))$ (in particular, $L = lk(\rho, K)$ if $H_i \cap H_j = \emptyset$). Now, L is an NH-ball or NH-sphere, since $\rho \in K$, and it is disconnected, since it contains the edges $\Delta(\{w_i, v_j\})$ and $\Delta(\{w_j, v_i\})$ in different components. The only possibility is that L is an NH-sphere of homotopy dimension 0 (see §2.2), but this cannot happen since there are two components of dimension at least one.

Assume now that $K = K(V, W, \mathcal{H})$ with the hypotheses as in the statement of (1). We will prove that K is a minimal NH-sphere by induction on d. The case $d = 0$ is trivial to check. Suppose $d \geq 1$. Let $\eta_i = (V \setminus \{v_i\}) \cup H_i$ $(1 \leq i \leq k+2)$ be the facets of K and note that $K = \eta_{k+2} + v_{k+2} * lk(v_{k+2}, K)$ since $\dim(\eta_{k+2}) = d$ and $|V_K| = d + 2$. By Lemma 2.2 and Theorem 3.5 it suffices to prove that $lk(v_{k+2}, K)$ is a minimal NH-sphere. But one can easily check that $lk(v_{k+2}, K)$ is isomorphic to $K(\tilde{V}, \tilde{W}, \tilde{\mathcal{H}})$ where $\tilde{V} = V \setminus \{v_{k+2}\}$, $\tilde{W} = H_{k+1}$ and $\tilde{\mathcal{H}} = \{H_1, \ldots, H_{k+1}\}$. The result then follows from the inductive hypothesis.

We next settle (2). Let K be a minimal NH-ball of dimension d. Then, there is a minimal NH-sphere S that admits a decomposition $S = K + L$. By (1) we know that $S = K(\tilde{V}, \tilde{W}, \tilde{\mathcal{H}})$ for some $\tilde{V} = \{v_1, \ldots, v_{k+2}\}$, $\tilde{W} = \{w_1, \ldots, w_{d-k}\}$ and $\tilde{\mathcal{H}} = \{H_1, \ldots, H_{k+2}\}$ satisfying $\emptyset = H_1 \subseteq H_2 \subseteq \cdots \subseteq H_{k+2} = W$. Let $\eta_{i_1}, \ldots, \eta_{i_q}$ be the facets of L, where $\eta_i = (V \setminus \{v_i\}) \cup H_i$ as above. Since by dimensional reasons $H_{i_1} = \cdots = H_{i_q} = \emptyset$ we can relabel the v_i's and H_i's so $i_j = j$ for $1 \leq j \leq q$. Then, $V := \tilde{V} \setminus \{v_1, \ldots, v_q\}$, $W := \tilde{W} \cup \{v_1, \ldots, v_q\}$ and $\mathcal{H} := \{H_{q+1} \cup \{v_1, \ldots, v_q\}, \ldots, H_{k+2} \cup \{v_1, \ldots, v_q\}\}$ satisfy the requirements of the statement.

The converse is similar to the case of minimal NH-spheres. □

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