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Lochs-type theorems beyond positive entropy

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Abstract

Lochs' theorem and its generalizations are conversion theorems that relate the number of digits determined in one expansion of a real number as a function of the number of digits given in some other expansion. In its original version, Lochs' theorem related decimal expansions with continued fraction expansions. Such conversion results can also be stated for sequences of interval partitions under suitable assumptions, with results holding almost everywhere, or in measure, involving the entropy. This is the viewpoint we develop here. In order to deal with sequences of partitions beyond positive entropy, this paper introduces the notion of log-balanced sequences of partitions, together with their weight functions. These are sequences of interval partitions such that the logarithms of the measures of their intervals at each depth are roughly the same. We then state Lochs-type theorems which work even in the case of zero entropy, in particular for several important log-balanced sequences of partitions of a number-theoretic nature.

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1 Introduction

Lochs' theorem [34] is a probabilistic statement about base changes. This conversion theorem relates, almost everywhere, the relative speed of approximation of decimal and regular continued fraction expansions to the quotient of the entropies of their respective dynamical systems. More generally, Lochs-type theorems amount to comparing the number of digits determined in one expansion of a real number as a function of the number of digits given in some other expansion.

Lochs' theorem and its extensions have given rise to a rich literature. The original setting of Lochs' theorem has been considered in [18] where an error term is provided based on the use of a Perron–Frobenius type operator; see also [19] for a central limit theorem, [20, 46] for the case of numbers x having a *Lévy's constant* $\lim_{n\to\infty}(\log q_n(x))/n$ (including the case of quadratic numbers), and [47] for an iterated logarithm law. Analogous results for the case of the beta-numeration and continued fractions have been established in [5, 22, 23, 32] and in [33] in the case of the beta-numeration and of the continued fractions for formal power series with coefficients in a finite field. More general transformations have then been considered in [10, 15, 16] for number theoretic fibred systems. Most of these results are stated in the case where both dynamical systems to be compared have positive entropy.

In the present paper we follow the viewpoint developed in [10, 15] where Lochs' theorem is understood as a way to see how decimal intervals fit into the fundamental intervals provided by the continued fraction expansion. Here also, and as stressed in [15], proofs are based on measure-theoretic covering arguments and not on the dynamics of specific maps. We introduce the notion of log-balanced sequences of partitions of the unit interval, inspired by [15] (where the terminology 'equipartition' is used). Roughly speaking, that a sequence $\mathcal{P} = \{P_n\}_{n\in\mathbb{N}}$ of partitions of the unit interval is log-balanced means that the intervals of each partition P_n have approximately the same measure $e^{-f(n)}$ as $n \to \infty$ (a.e. or in measure), where f is called a weight function (see Definition 2.1). We then deal with the Lochs index, a now classical parameter in base changing, denoted here by L_n and defined for two sequences of partitions as follows. Let $\mathcal{P}^1 = \{P_n^1\}_{n\in\mathbb{N}}$ and $\mathcal{P}^2 = \{P_n^2\}_{n\in\mathbb{N}}$ be two sequences of partitions of the unit interval. For each $i = 1, 2$, let E^i be the set of endpoints of all the intervals of the partitions in $\mathcal{P}^i$ and, for each $x \in [0, 1] \setminus E^i$, let $I_n^i(x)$ be the interval of P_n^i containing x. Then, the Lochs index is defined by

$$L_n(x, \mathcal{P}^1, \mathcal{P}^2) := \sup\{\ell \in \mathbb{N} \mid I_n^1(x) \subseteq I_\ell^2(x)\}, \tag{1}$$

for each $x \in [0, 1]\setminus(E^1 \cup E^2)$. (Strictly speaking, we work with *topological* partitions, see Sect. 2.1.)

The behavior (a.e. or in measure) of Lochs' index is usually described in the literature in terms of entropy (we discuss the notion of entropy in Sects. 2.1 and 4.4). Our work is inspired by the following result, which is Theorem 4 from [15]. Let λ be a Borel probability measure on [0, 1]. If $\mathcal{P}^1$ and $\mathcal{P}^2$ are sequences of partitions having positive a.e. entropies h_1 and h_2 with respect to λ, then

$$\lim_{n\to\infty} \frac{L_n(x, \mathcal{P}^1, \mathcal{P}^2)}{n} = \frac{h_1}{h_2} \quad a.e.\ (\lambda).$$

As Dajani and Fieldsteel [15], we also consider a variant in measure of the above result.

Our main results, Theorems 6.8 and 6.14, work as follows. Let $\mathcal{P}^1$ and $\mathcal{P}^2$ be two log-balanced sequences of partitions with weight functions f_1 and f_2, both of which either a.e. or in measure. We give sufficient conditions in order to ensure that

$$\lim_{n\to\infty} \frac{f_2(L_n(x, \mathcal{P}^1, \mathcal{P}^2))}{f_1(n)} = 1 \tag{2}$$

either a.e. or in measure (λ), respectively.

Our results extend the results of [15], by providing asymptotic relations as (2) for partitions which may not have positive entropy. The assumptions of our main results regard separately the weight functions f_1 and f_2.

For an illustration, consider the sequence of partitions given by the binary numeration, i.e., $\mathcal{P}^1 = \mathcal{B} = \{B_n\}_{n\in\mathbb{N}}$ with

$$B_n = \left\{ \left(0, \frac{1}{2^n}\right), \left(\frac{1}{2^n}, \frac{2}{2^n}\right), \ldots, \left(\frac{2^n-1}{2^n}, 1\right) \right\},$$

and the Farey sequence of partitions $\mathcal{P}^2 = \mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ (the latter of which has zero entropy, see Sect. 4.6), where each F_n is determined by the set $E_n^{\mathcal{F}}$ of endpoints of its intervals, given as follows:

$$E_n^{\mathcal{F}} := \left\{ \frac{p}{q} : \ p, q \geq 1, \ \gcd(p, q) = 1 \text{ and } q \leq n+1 \right\}.$$

Our results imply that, with respect to the Lebesgue measure,

$$\lim_{n\to\infty} \frac{2 \log L_n(x, \mathcal{B}, \mathcal{F})}{(\log 2) n} = 1 \quad \text{a.e.}$$

(where by log we mean the natural logarithm). Note that a change of scale is performed on the Loch's index $L_n(x, \mathcal{B}, \mathcal{F})$ which occurs as $\log L_n(x, \mathcal{B}, \mathcal{F})$ compared to the index n.

Our main examples of sequences of partitions (see Sect. 4) are of an arithmetic nature. They are associated with numeration systems (see Sect. 4.2), dynamically, with fibred systems (see Sect. 4.3), and more specifically with continued fractions with the Stern–Brocot tree (see Sect. 4.5), the Farey sequence (see Sect. 4.6), and a partition related to the three-distance theorem (see Sect. 4.7).

The study of base changes and of the Lochs' index opens a large scope of potential applications. Indeed, sequences of partitions can model numeration systems (see Sect. 4.2), as well as sources for the production of digits; as an example, consider the dichotomic selection on words for selection algorithms developed in [1] and see also Sect. 7. A further motivation is the dynamic generation of characteristic Sturmian words of uniform random parameters (see Sect. 7 for more on this topic).

Plan of the article

The key notion of a log-balanced sequence of partitions is introduced in Sect. 2.1 together with the weight functions for our main examples of sequences of partitions. The definition of the Lochs' index is given in Sect. 2.2 which also presents the main theorems of this paper. Section 3 gives some basic results on log-balanced sequences of partitions. The examples of sequences of partitions considered in the present paper are detailed in Sect. 4, with in particular the notion of a numeration system given by a sequence of partitions in Sect. 4.2 and a discussion on fibred systems in Sect. 4.3. For the case of conversions between the Farey and the continued fraction sequences of partitions, explicit formulas for the corresponding Lochs' indexes are given in Sect. 5. As a consequence, we derive in particular a Lochs-type theorem for the conversion from the continued fraction to the Farey sequences of partitions in law (see Sect. 5.1). The proofs of our main general conversion results are given in Sect. 6. We conclude this paper with open questions and by developing connections with sources and tries in Sect. 7.

2 Log-balancedness and Lochs-type theorems

The aim of this section is to introduce the main notions and state our main results. In Sect. 2.1, we define the notions of log-balancedness and weight function. As far as we know, these notions are introduced in this paper for the first time in the context of Lochs' index. We also relate weight functions to the entropies of sequences of partitions when they are positive. Theorem 2.4 presents our main examples of log-balanced sequences of partitions together with their weight functions. In Sect. 2.2, we formally define the notion of Lochs' index, following the classical approach of the literature. Finally, in the same section, we state our main results.

2.1 Log-balanced sequences of partitions

We first fix some general notation and definitions. We consider that $0 \in \mathbb{N}$. The notation $\overline{A}$ stands for the usual topological closure of A. The Lebesgue measure is denoted by $|\cdot|$.

A *topological partition of* [0, 1] is a collection of pairwise disjoint open nonempty intervals so that the union of their closures equals [0, 1]. Notice that such a collection is necessarily at most countable.

By a *sequence of partitions* $\mathcal{P} = \{P_n\}_{n\in\mathbb{N}}$, we mean a sequence of topological partitions of [0, 1]. We refer to the intervals in P_n as the *intervals at depth* n of $\mathcal{P}$. We say that $\mathcal{P}$ is *self-refining* if, for each $n \in \mathbb{N}$, each interval of P_{n+1} is contained in (and possibly equal to) an interval of P_n. A sequence of partitions $\mathcal{P} = \{P_n\}_{n\in\mathbb{N}}$ is *strictly self-refining* if it is self-refining and $P_{n+1} \neq P_n$ for each $n \in \mathbb{N}$; or, equivalently, for each $n \in \mathbb{N}$, there exists an interval I_n in P_n which is the union of at least two different intervals in P_{n+1}. We denote by $E^{\mathcal{P}}$ the set of endpoints of the intervals in the partitions in $\mathcal{P}$.

Given a sequence of partitions $\mathcal{P}$, $n \in \mathbb{N}$, and $x \in [0, 1] \setminus E^{\mathcal{P}}$, we denote by $I_n^{\mathcal{P}}(x)$ the unique interval in P_n to which x belongs. If there is no risk of ambiguity, we denote $I_n^{\mathcal{P}}(x)$ simply by $I_n(x)$ and $E^{\mathcal{P}}$ simply by E.

Definition 2.1 Let λ be a Borel probability measure on [0, 1]. Let $\mathcal{P}$ be a sequence of partitions. We say that $\mathcal{P}$ is *log-balanced a.e.* (resp. *in measure*) with respect to λ if $\lambda(E) = 0$ and there is some function $f : \mathbb{N} \to \mathbb{R}$ such that $f(n) \to +\infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{-\log \lambda(I_n(x))}{f(n)} = 1 \quad a.e. \text{ (resp. in measure)} \quad (\lambda).$$

If so, f is called a *weight function of* $\mathcal{P}$ *a.e.* (resp. *in measure*) with respect to λ.

The weight function is not unique. If f is a weight function for a sequence of partitions $\mathcal{P}$ with respect to some measure λ, any other function $g : \mathbb{N} \to \mathbb{R}$ so that $f(n)/g(n) \to 1$ as $n \to \infty$ is also a weight function with respect to λ. In the particular case of an a.e. log-balanced self-refining system of partitions, the weight function might be chosen to be nondecreasing. This is proved in Proposition 3.5.

When, in the above definition, the weight f is linear, one recovers the following notions of entropy, as given in [15], which are reminiscent of the Shannon–McMillan–Breiman theorem (see Theorem 4.4).

Definition 2.2 Let $\mathcal{P} = \{P_n\}_{n\in\mathbb{N}}$ be a sequence of partitions and λ be a Borel probability measure λ on the unit interval. We say $\mathcal{P}$ *has entropy* h *a.e. (resp. in measure) with respect to* λ if $\lambda(E) = 0$ and

$$\lim_{n\to\infty} \frac{-\log \lambda(I_n(x))}{n} = h \quad \text{a.e. (resp. in measure) } (\lambda). \tag{3}$$

Remark 2.3 The assumption $\lambda(E) = 0$ is not given in [15, Definition 2]; nevertheless, it is implicit in it. The reason is that in that work, the partitions are made up of semi-open intervals in such a way that for any $x \in [0, 1)$ and each $n \in \mathbb{N}$ there is a well defined $I_n(x)$ in P_n such that $x \in I_n(x)$. Thus, in that context, (3) implies $\lambda(\{x\}) = 0$ for every $x \in [0, 1)$ (cf. Proposition 3.2 of this work). Notice also that in [15], $\lambda([0, 1)) = 1$.

2.1.1 Main examples of weight functions

Our main examples of sequences of partitions are detailed in Sect. 4. It includes number-theoretic fibred systems, especially the sequence of partitions associated with continued fractions and the binary numeration system. We also study the Farey sequence of partitions given by the classical Farey sequence. We are interested in it because of its strong relation with Sturmian words (see e.g. [7]). In addition, we study the Stern–Brocot sequence of partitions associated with the Stern–Brocot tree. Finally, we also study what we call three-distance sequences of partitions parameterized by an irrational $\alpha \in (0, 1)$, associated with the family of Kronecker-Weyl sequences $n \mapsto \{n\alpha\}$. The precise definitions of all of these sequences of partitions are given throughout Sect. 4. Also throughout that section, the log-balancedness of each of these sequences is analyzed, thus obtaining the results summarized in the following theorem.

Theorem 2.4 *The following assertions hold with respect to the Lebesgue measure:*

(i) The binary sequence of partitions $\mathcal{B}$ is a.e. log-balanced with weight function

$$f_{\mathcal{B}}(n) = (\log 2)n.$$

(ii) The continued fraction sequence of partitions $\mathcal{CF}$ is a.e. log-balanced with weight function

$$f_{\mathcal{CF}}(n) = \frac{\pi^2}{6\log 2}n.$$

(iii) More generally, if $\mathcal{H}$ is any sequence of partitions having positive entropy h a.e. (resp. in measure), then $\mathcal{H}$ is log-balanced a.e. (resp. in measure) with weight function

$$f_{\mathcal{H}}(n) = hn.$$

(iv) The Farey sequence of partitions $\mathcal{F}$ is a.e. log-balanced with weight function

$$f_{\mathcal{F}}(n) = 2\log n \quad \textit{for each } n \geq 1.$$

(v) The three-distance sequence of partitions $3\mathcal{D}(\alpha)$ is a.e. log-balanced, for a.e. $\alpha \in (0, 1)$, with weight function

$$f_{3\mathcal{D}(\alpha)}(n) = \log n \quad \textit{for each } n \geq 1.$$

However, there is an uncountable set of numbers α for which $3\mathcal{D}(\alpha)$ is not even log-balanced in measure.

(vi) The Stern–Brocot sequence of partitions $\mathcal{SB}$ is log-balanced in measure with weight function

$$f_{\mathcal{SB}}(n) = \frac{\pi^2}{6}\frac{n}{\log n} \quad \textit{for each } n \geq 2.$$

Nevertheless, the $\mathcal{SB}$ sequence of partitions is not log-balanced a.e.

The fact that we work with the logarithm of the measures of the intervals (and not with the measures themselves) allows for some non-uniformity for the measures of the intervals. This is illustrated in particular by the case of the Stern–Brocot sequence of partitions, which is log-balanced in measure (but not a.e.) with respect to the Lebesgue measure.

2.2 Lochs-type theorems

We start by giving a precise definition of Lochs' index, following the approach of [10] and [15]. If $\mathcal{P}^1$ and $\mathcal{P}^2$ are sequences of partitions, then, for each $i = 1, 2$, we denote by E^i the set of endpoints of all the intervals of the partitions in $\mathcal{P}^i$ and, for each $x \in [0, 1] \setminus E^i$, $I_n^i(x)$ the interval of P_n^i containing x.

Definition 2.5 If $\mathcal{P}^1$ and $\mathcal{P}^2$ are sequences of partitions and $n \in \mathbb{N}$, the *Lochs' index* is defined as

$$L_n(x, \mathcal{P}^1, \mathcal{P}^2) := \sup\{\ell \in \mathbb{N} \mid I_n^1(x) \subseteq I_\ell^2(x)\}, \quad \text{for each } x \in [0, 1] \setminus (E^1 \cup E^2).$$

When both sequences of partitions involved are log-balanced a.e. or in measure with respect to the same Borel probability measure λ, the Lochs' index is a.e. finite with respect to λ. This will be proved in Proposition 3.3.

2.2.1 Results for general sequences of partitions

We first recall results from the literature. Let us consider the sequence of partitions $\mathcal{P}^1 = \mathcal{D} = \{D_n\}$ of the decimal numeration system, i.e.,

$$D_n = \left\{\left(0, \frac{1}{10^n}\right), \left(\frac{1}{10^n}, \frac{2}{10^n}\right), \ldots, \left(\frac{10^n - 1}{10^n}, 1\right)\right\},$$

and let $\mathcal{P}^2$ be the classical continued fraction sequence of partitions $\mathcal{CF}$ (as given in Sect. 4.4.4). The following is the classical Lochs' theorem.

Theorem 2.6 ([34]) *The following holds with respect to the Lebesgue measure:*

$$\lim_{n\to\infty} \frac{L_n(x, \mathcal{D}, \mathcal{CF})}{n} = \frac{6 \log 2 \log 10}{\pi^2} \quad \text{a.e.}$$

Dajani and Fieldsteel [15] proved theorems like the above one but in a more general setting, involving two sequences of partitions $\mathcal{P}_1$ and $\mathcal{P}_2$, both of positive entropies a.e. or in measure. Their results are as follows.

Theorem 2.7 ([15, Theorem 4]) *If $\mathcal{P}^1$ and $\mathcal{P}^2$ are sequences of partitions having positive a.e. entropies h_1 and h_2 with respect to some Borel probability measure λ on*

[0, 1], *then*

$$\lim_{n\to\infty} \frac{L_n(x, \mathcal{P}^1, \mathcal{P}^2)}{n} = \frac{h_1}{h_2} \quad a.e.\ (\lambda).$$

Theorem 2.8 ([15, Theorem 6]) *If $\mathcal{P}^1$ and $\mathcal{P}^2$ are sequences of partitions having positive entropies h_1 and h_2 in measure with respect to some Borel probability measure λ on* [0, 1] *and $\mathcal{P}^2$ is self-refining, then*

$$\lim_{n\to\infty} \frac{L_n(x, \mathcal{P}^1, \mathcal{P}^2)}{n} = \frac{h_1}{h_2} \quad \textit{in measure}\ (\lambda).$$

We now state our main results which are extensions of the above theorems to log-balanced sequences of partitions.

Theorem 6.8 *Let $\mathcal{P}^1$ and $\mathcal{P}^2$ be two a.e. log-balanced sequences of partitions with weight functions f_1 and f_2, respectively, with respect to some Borel probability measure λ on* [0, 1] *such that all the following assertions hold:*

(i) $\sum_{n=1}^{\infty} e^{-\delta f_1(n)} < \infty$ *for every* $\delta > 0$;
(ii) f_2 *is nondecreasing;*
(iii) $f_2(n+1) - f_2(n) = o(f_2(n))$ *as* $n \to \infty$.

Then,

$$\lim_{n\to\infty} \frac{f_2(L_n(x, \mathcal{P}^1, \mathcal{P}^2))}{f_1(n)} = 1 \quad a.e.\ (\lambda).$$

Remark 2.9 The assumptions of this theorem can be easily checked in natural instances.

1. Assumption (i) above holds for any function f satisfying that $\lim_{n\to\infty} f(n)/\log n = \infty$. In contrast, it does not hold if $\limsup_{n\to\infty} f(n)/\log n$ is finite.
2. If $\mathcal{P}_2$ is self-refining, then the weight function f_2 can be chosen to be nondecreasing (see Proposition 3.5).
3. A weight function f_2 satisfies assumption (iii) above if and only if $f_2(n+1)/f_2(n) \to 1$ as $n \to \infty$ or, equivalently, $\sqrt[n]{|f_2(n)|} \to 1$ as $n \to \infty$.

Theorem 6.14 *Let $\mathcal{P}^1$ and $\mathcal{P}^2$ be two sequences of partitions that are log-balanced in measure, having respective weight functions f_1 and f_2, with respect to some Borel probability measure λ on* [0, 1] *such that all the following assertions hold:*

(i) $\mathcal{P}^2$ *is self-refining;*
(ii) f_2 *is nondecreasing;*
(iii) $f_2(n+1) - f_2(n) = o(f_2(n))$ *as* $n \to \infty$.

Then,

$$\lim_{n\to\infty} \frac{f_2(L_n(x; \mathcal{P}^1, \mathcal{P}^2))}{f_1(n)} = 1 \quad \textit{in measure}\ (\lambda).$$

Remark 2.10 In this case, there are no assumptions on f_1 apart from it being a weight function of $\mathcal{P}^1$ with respect to λ. Possible assumptions equivalent to assertion (iii) are given in Remark 2.9 above.

The above theorems are proved in Sect. 6. To prove them, we deal with the corresponding superior and inferior limits separately. This gives four propositions. The conditions in their statements are weaker than those asked in Theorems 6.8 and 6.14 but, for the propositions regarding the inferior limits, are also more technical. The first two, Propositions 6.6 and 6.7, consider a.e. convergence and prove Theorem 6.8. Propositions 6.12 and 6.13 prove the result in measure, namely Theorem 6.14.

2.2.2 Positive entropy versus Farey and Stern–Brocot

We focus here on conversion results involving the Farey and the Stern–Brocot sequences of partitions.

In the case $\mathcal{P}^2$ is the Farey sequence of partitions $\mathcal{F}$ (see Sect. 4.6) and λ is the Lebesgue measure, Theorem 6.8 applies when $\mathcal{P}^1$ is any sequence of partitions $\mathcal{H}$ of positive entropy h a.e. and yields the following:

$$\lim_{n\to\infty} \frac{\log L_n(x, \mathcal{H}, \mathcal{F})}{n} = \frac{h}{2} \quad \text{a.e.}, \tag{4}$$

with respect to the Lebesgue measure. In particular, if $\mathcal{H}$ is the binary sequence of partitions $\mathcal{B}$ or the continued fraction sequence of partitions $\mathcal{CF}$, then h is $\log 2$ or $\pi^2/(6\log 2)$, respectively. Note that this result admits a nice interpretation in terms of Sturmian words: because of the connection between Sturmian words and the Farey sequences (see Sect. 4.6.2), the assertion for $\mathcal{H} = \mathcal{B}$ means that the first n digits in the binary expansion of some $x \in [0, 1]$ give the prefix of length roughly $2^{n/2}$ of the characteristic Sturmian word associated with x. Concerning the case $\mathcal{H} = \mathcal{CF}$, note also that using classical results about the statistics of continued fraction expansions, a normal law is even proved in Theorem 5.4 for the Lochs' index $L_n(x, \mathcal{CF}, \mathcal{F})$.

Notice that assumption (i) of Theorem 6.8 is not satisfied in the case where $\mathcal{P}^1 = \mathcal{F}$, because the corresponding weight function is $f_{\mathcal{F}}(n) = 2\log n$ (see Theorem 2.4 and Remark 2.9). In Sect. 5.6, we show, by alternative means, that

$$\lim_{n\to\infty} \frac{L_n(x, \mathcal{F}, \mathcal{CF})}{\log n} = \frac{12\log 2}{\pi^2} \quad \text{a.e.},$$

with respect to the Lebesgue measure. In this case, our general results only provide convergence in measure. The fact that we may obtain a stronger convergence is not surprising in such a case: the two partitions, continued fractions and Farey, are strongly related in their constructions (see e.g., Lemma 5.2), while the proofs of the general results are based on the comparison of the growth of the measures of the intervals.

Lastly, the case of the Stern–Brocot sequence of partitions and positive entropy is an example of direct application of Theorem 6.14 together with Theorem 2.4. If $\mathcal{H}$ is any sequence of partitions of positive entropy h in measure, then

$$\lim_{n\to\infty} \frac{L_n(x, \mathcal{SB}, \mathcal{H})}{n \log L_n(x, \mathcal{SB}, \mathcal{H})} = \frac{6h}{\pi^2} \quad \text{in measure}$$

and

$$\lim_{n\to\infty} \frac{L_n(x, \mathcal{H}, \mathcal{SB})}{n/\log n} = \frac{\pi^2}{6h} \quad \text{in measure,}$$

with respect to the Lebesgue measure.

3 Basic properties of log-balanced sequences of partitions

This section gathers basic results which describe the behavior of log-balanced sequences of partitions and of Lochs' index. The first one, namely Proposition 3.1, shows that the measures of intervals of a log-balanced sequence of partitions tend to zero as their depths tend to infinity. This has two consequences: first, points have zero measure (Proposition 3.2); second, Lochs' index is a.e. finite (Proposition 3.3). Proposition 3.4 then shows that if a sequence of partitions is log-balanced with respect to a measure λ, so is it with respect to any measure equivalent to λ. This property is relevant for the study of number-theoretic fibred systems (see Sect. 4.3). Proposition 3.5 shows that any a.e. log-balanced sequence of partitions admits some nondecreasing weight function. Finally, Proposition 3.6 proves that a sequence of partitions with a sub-exponential number of intervals at each depth has zero entropy. In particular, if it is log-balanced, its weight function is in $o(n)$. This is the case for the Farey and the three-distance sequences of partitions.

The following proposition shows that the norm of the partitions of a log-balanced sequence tends to zero.

Proposition 3.1 *Let $\mathcal{P} = \{P_n\}_{n\in\mathbb{N}}$ be a sequence of partitions. If $\mathcal{P}$ is log-balanced in measure (or even a.e.) with respect to some Borel probability measure λ, then*

$$\|P_n\|_\lambda := \sup_{I\in P_n} \lambda(I) \to 0 \quad \text{as } n \to \infty.$$

Proof Suppose otherwise $\|P_n\|_\lambda \not\to 0$ as $n \to \infty$. Thus, there is some $c > 0$ and some increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ such that $\|P_{n_k}\|_\lambda \to c$ as $k \to \infty$. Let $J_k \in P_{n_k}$ such that $\lambda(J_k) > c/2$, for large enough k.

Let f be a weight function of $\mathcal{P}$ in measure (or even a.e.) with respect to λ. Since $f(n_k) \to +\infty$ as $k \to \infty$, we have that $1+\log(c/2)/f(n_k) \geq 1/2$ for k large enough. Hence, as for each $x \in J_k$, we have $J_k = I_{n_k}(x)$, then

$$\frac{c}{2} < \lambda(J_k) \leq \lambda\left(\left\{x \in [0,1] : 1 + \frac{\log\lambda(I_{n_k}(x))}{f(n_k)} > 1 + \frac{\log(c/2)}{f(n_k)}\right\}\right)$$

$$\leq \lambda \left(\left\{ x \in [0,1] : \left| 1 + \frac{\log \lambda(I_{n_k}(x))}{f(n_k)} \right| > \frac{1}{2} \right\} \right),$$

for large enough k. This shows that $-\log(\lambda(I_n(x))/f(n)$ does not converge towards 1 as $n \to \infty$ in measure and thus neither does a.e. This contradicts that f is a weight function of $\mathcal{P}$. □

The next proposition shows that the property of being log-balanced with respect to λ implies that λ charges no point.

Proposition 3.2 *Let $\mathcal{P} = \{P_n\}_{n\in\mathbb{N}}$ a sequence of partitions. If $\mathcal{P}$ is log-balanced in measure (or even a.e.) with respect to some Borel probability measure λ, then $\lambda(\{x\}) = 0$ for each $x \in [0,1]$.*

Proof On the one hand, $\lambda(E) = 0$ by Definition 2.1. On the other hand, if $x \in [0,1]\setminus E$ and $n \in \mathbb{N}$, then $\lambda(\{x\}) \leq \lambda(I_n(x)) \leq \|P_n\|_\lambda$ and thus $\lambda(\{x\}) = 0$ by Proposition 3.1. □

In our setting of log-balanced sequences of partitions, Lochs' index takes finite values a.e. such as stated below, where we recall that E^i stands for the set of endpoints of the partition $\mathcal{P}^i$, $i = 1, 2$.

Proposition 3.3 *Let $\mathcal{P}^1$ and $\mathcal{P}^2$ be sequences of partitions. If $\mathcal{P}^2$ is log-balanced a.e. (or in measure) with respect to some Borel probability measure λ and $\lambda(E^1) = 0$, then the Lochs' index $L_n(x, \mathcal{P}^1, \mathcal{P}^2)$ is finite a.e. with respect to λ for each $n \in \mathbb{N}$.*

Proof Let $n \in \mathbb{N}$. Since $\mathcal{P}^2$ is log-balanced a.e. (or in measure), $\lambda(E^2) = 0$. Let J be the union of all the intervals I_n^1 in P_n^1 such that $\lambda(I_n) = 0$. Clearly, $\lambda(J) = 0$. Hence, it suffices to prove that $L_n(x, \mathcal{P}^1, \mathcal{P}^2)$ is finite for every $x \in [0,1] \setminus (E^1 \cup E^2 \cup J)$. For that purpose, let $x \in [0,1] \setminus (E^1 \cup E^2 \cup J)$. By Proposition 3.1, $\lambda(I_\ell^2(x)) \to 0$ as $\ell \to \infty$. As $\lambda(I_n^1(x)) > 0$, there must be some $\ell \in \mathbb{N}$ such that $I_n^1(x) \not\subseteq I_\ell^2(x)$ and thus $L_n(x, \mathcal{P}_1, \mathcal{P}^2)$ is finite. □

Equivalent measures yield the same log-balanced sequences and weight functions.

Proposition 3.4 *Let $\mathcal{P} = \{P_n\}_{n\in\mathbb{N}}$ be a sequence of partitions that is log-balanced a.e. (resp. in measure) with respect to some Borel probability measure λ. If μ is another Borel probability measure equivalent to λ (i.e., both $d\mu/d\lambda$ and $d\lambda/d\mu$ are bounded), then $\mathcal{P}$ is also log-balanced a.e. (resp. in measure) with respect to μ with the same weight function.*

Proof By Proposition 3.1, $\mu(I_n(x)) \to 0$ as $n \to \infty$ for every $x \in [0,1] \setminus E$. Now the proof can be completed along the same lines as in the proof of [10, Lemma 2.4] in order to prove that $\lim_{n\to\infty} \frac{\log \lambda(I_n(x))}{\log \mu(I_n(x))} = 1$ a.e. □

For the validity of Theorem 6.8, we assume that the weight f_2 of $\mathcal{P}^2$ is nondecreasing. The following lemma shows that this assumption does not per se rules out any particular a.e. log-balanced self-refining sequence $\mathcal{P}^2$.

Proposition 3.5 *Every self-refining a.e. log-balanced sequence of partitions $\mathcal{P}$ with respect to some Borel probability measure λ admits some nondecreasing a.e. weight function with respect to λ.*

Proof Let f be an a.e. weight function of $\mathcal{P}$. Let $x \in [0,1] \setminus E$ for which $-\log \lambda(I_n(x))/f(n) \to 1$ as $n \to \infty$. Let $g(n) = -\log \lambda(I_n(x))$. By construction, $g(n)/f(n) \to 1$ as $n \to \infty$. Hence, as f is an a.e. weight function of $\mathcal{P}$, so is g. Moreover, since $\mathcal{P}$ is self-refining, the definition of g implies that g is nondecreasing. $\square$

The next result shows that a sequence of partitions with polynomially many intervals at each depth has zero entropy a.e.

Proposition 3.6 *Let $\mathcal{P} = \{P_n\}_{n\in\mathbb{N}}$ be a sequence of partitions and let λ be a Borel probability measure on $[0,1]$. Suppose that the number $\#(P_n)$ of intervals at depth n is such that $\log \#(P_n) = o(n)$ as $n \to \infty$. Then, $\mathcal{P}$ has zero entropy a.e.*

Proof Notice that $\log \#(P_n) = o(n)$ implies $\limsup_{n\to\infty} \#(P_n)^{1/n} = 1$. Thus, by the root test, $\sum_{n=0}^{\infty} \#(P_n) e^{-\epsilon n} < \infty$ for each $\epsilon > 0$.

Let $h_n(x) := -\log \lambda(I_n(x))/n$. And, for each $\epsilon > 0$ and $n \in \mathbb{N}$, let

$$A_{n,\epsilon} = \{x \in [0,1] \setminus E : h_n(x) \geq \epsilon\} = \{x \in [0,1] \setminus E : \lambda(I_n(x)) \leq e^{-\epsilon n}\}\,.$$

Therefore,

$$\lambda(A_{n,\epsilon}) \leq \sum_{I_n \in \mathcal{P}_n :\, I_n \cap A_{n,\epsilon} \neq \emptyset} \lambda(I_n) \leq \#(P_n) e^{-\epsilon n}\,.$$

Since $\sum_{n=0}^{\infty} \#(P_n) e^{-\epsilon n} < \infty$, the Borel–Cantelli Lemma implies

$$\lambda(\{x : x \in A_{n,\epsilon} \text{ i.o.}\}) = 0.$$

Thus, $x \notin A_{n,\epsilon}$ for n large enough a.e., that is, $\lim_{n\to\infty} h_n(x) = 0$ a.e. $\square$

4 Main examples

In this section, we introduce our main examples of sequences of partitions and discuss their log-balancedness. Throughout this section we prove Theorem 2.4.

This section is organized as follows. In Sect. 4.1, we show that one can give an explicit log-balanced sequence of partitions with weight f for any given function f that tends to infinity. In Sect. 4.2, we make explicit the connection between sequences of partitions and numeration systems. In Sect. 4.3, we build log-balanced sequences of partitions from fibred systems. In Sect. 4.4, we consider fibred numeration systems with positive entropy, which includes the sequences of partitions corresponding to the binary numeration system, the beta-numeration, and continued fraction expansions. In Sect. 4.5, we introduce the Stern–Brocot sequence of partitions, which is induced by a fibred system of zero entropy. In Sect. 4.6, we present an example of zero-entropy sequences of partitions not induced by fibred systems, namely the Farey sequence of partitions. Lastly, in Sect. 4.7, we discuss a sequence of partitions associated with the three-distance theorem.

4.1 A realization result

The next proposition shows that for any $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ so that $f(n) \to \infty$ as $n \to \infty$ there exists a sequence of partitions with f as an a.e. weight function with respect to the Lebesgue measure.

Proposition 4.1 *Let $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$, with $\lim_{n\to\infty} f(n) = \infty$. Then, there exists an a.e. log-balanced sequence of partitions $\mathcal{P} = \{P_n\}_{n\in\mathbb{N}}$ with weight function f, with respect to the Lebesgue measure. Moreover, if f is nondecreasing, then $\mathcal{P}$ is self-refining. Furthermore, if $n \mapsto \lfloor e^{f(n)} \rfloor$ is (strictly) increasing, then $\mathcal{P}$ is strictly self-refining.*

Proof We build $\mathcal{P}$ by giving for each $n \in \mathbb{N}$ the set endpoints of the intervals of P_n. For that purpose, we take as a reference the sets $E_k^{\mathcal{B}} = \{a/2^k : a \in \mathbb{N} \text{ and } 0 \leq a \leq 2^k\}$ of endpoints of the binary partition at depth k.

Fix $n \in \mathbb{N}$ and consider the only integer $k \geq 0$ so that $2^k \leq e^{f(n)} < 2^{k+1}$. These relations allows one to perform a change of scale between indices of depths k and n. Let P_n be the topological partition of $[0, 1]$ whose endpoint set is

$$E_n = E_k^{\mathcal{B}} \cup \{a/2^{k+1} : a = 2i + 1 \text{ and } 0 \leq i \leq \lfloor e^{f(n)} \rfloor - 2^k - 1, i \in \mathbb{N}\}.$$

Notice that $E_k^{\mathcal{B}} \subseteq E_n \subsetneq E_{k+1}^{\mathcal{B}}$ because E_n is $E_k^{\mathcal{B}}$ together with the leftmost $\lfloor e^{f(n)} \rfloor - 2^k$ endpoints of $E_{k+1}^{\mathcal{B}}$. Hence, if $n \mapsto \lfloor e^{f(n)} \rfloor$ is nondecreasing, then the sequence of partitions $\{P_n\}$ is self-refining, and if $n \mapsto \lfloor e^{f(n)} \rfloor$ is increasing, then $P_n \subsetneq P_{n+1}$.

Let $x \notin E^{\mathcal{B}} := \bigcup_{k=0}^{\infty} E_k^{\mathcal{B}}$. Then, by construction, $2^{-k-1} \leq |I_n(x)| \leq 2^{-k}$, that is,

$$\frac{1}{2e^{f(n)}} \leq |I_n(x)| < \frac{2}{e^{f(n)}}.$$

The fact that $f(n) \to \infty$ as $n \to \infty$ implies that

$$\lim_{n\to\infty} \frac{-\log |I_n(x)|}{f(n)} = 1 \quad a.e.$$

□

Remark 4.2 Notice that, in the above proposition, if $f(n)/n \to 0$ (resp. $f(n)/n \to \infty$) as $n \to \infty$, then the sequence of partitions $\mathcal{P}$ has 0 (resp. ∞) a.e. entropy with respect to the Lebesgue measure.

4.2 From sequences of partitions to numeration systems

By numeration system given by partitions, we mean a system of representation of numbers using sequences of labels, and even of digits in most of the cases, provided by a sequence of partitions $\mathcal{P} = \{P_n\}_{n\in\mathbb{N}}$ endowed with labels. This involves in particular the most classical numerations for real numbers, such as b-ary representations, as well as representations based on continued fractions.

A *numeration system by a log-balanced sequence of partitions* over the (at most countable) alphabet $\mathcal{A}$, denoted as $\mathcal{N} = (\mathcal{P}, \rho)$, is defined by a strictly self-refining sequence of partitions $\mathcal{P} = \{P_n\}_{n\in\mathbb{N}}$ with $P_0 = \{(0, 1)\}$, assumed to be log-balanced with respect to the Lebesgue measure, together with a sequence of labeling functions $\rho = \{\rho_n\}_{n\geq 1}$, with $\rho_n : P_n \to \mathcal{A}$ for all n, that satisfies the following condition.

(H) For each $I \in \mathcal{P}_n$ and each distinct $J, J' \in P_{n+1}$ such that $J, J' \subset I$, $\rho_{n+1}(J) \neq \rho_{n+1}(J')$. In other words, the restriction of the labeling map ρ_{n+1} is injective when restricted to the set $P_{n+1}|_I$ of all intervals $J \in P_{n+1}$ contained in $I \in P_n$.

In particular, by Proposition 3.1, one has $\|P_n\| = \sup\{|I| : I \in P_n\}$ tends to 0 as $n \to \infty$. The sequence of partitions is thus *generating* (i.e., for a.e. $x, y \in [0, 1]$, if $x \neq y$, then there exist $n \in \mathbb{N}$, $P, Q \in P_n$, and $P \neq Q$, such that $x \in P$ and $y \in Q$.)

In full generality, according e.g. to the formalism developed in [4], a *numeration system* is a triple $(X, \mathcal{A}, \varphi)$, where X is a set (the set of elements to be represented), $\mathcal{A}$ a finite or countable set (the alphabet of the representation), and φ an injective map $\varphi\colon X \to \mathcal{A}^{\mathbb{N}}$. Accordingly, given a sequence of partitions $\mathcal{P}$ that satisfies the above conditions, we want to define a map that associates with every x in $[0, 1]$ the sequence of labels of intervals to which x belongs. Such a map is well-defined on points that do not belong to the set E of endpoints of the intervals of the partitions. In any case, the set E is a countable set[1]. Hence here X is equal to $[0, 1] \setminus E$ and this gives a coding map φ that associates with an element of $[0, 1] \setminus E$ the sequence of labels of the intervals to which it belongs:

$$\varphi : [0, 1] \setminus E \to \mathcal{A}^{\mathbb{N}},\ x \mapsto \rho_1(I_1(x))\rho_2(I_2(x)) \dots \text{ where } x \in I_n(x) \text{ for all } n.$$

Let us check that the map φ is injective. Let $x, y \in X$ and suppose $\varphi(x) = \varphi(y)$. Observe that x and y both belong to $[0, 1]$. Let us prove by induction that x and y belong to the same interval $I_n \in P_n$ for all n. Assume that the induction property holds for some positive n. The points x and y thus belong to the same interval $I_n \in P_n$, and by Condition (H), x, y belong to the same $I_{n+1} \in P_{n+1}$, which ends the induction proof. Thus, $x, y \in \bigcap_{n\geq 1} I_n$. Since $\{I_n\}$ is a nested sequence of intervals whose length tends to 0, $\bigcap_{n\geq 1} I_n$ has only one element. Hence, $x = y$. We thus get a numeration system as defined above. In other words, by assumption (H), we have for any $x \in [0, 1] \setminus E$ that $\{x\} = \bigcap_{n\geq 1} \overline{I_n(x)}$ and the sequence of labels $\{\rho_n(I_n(x))\}_{n\geq 1}$ encodes x univocally. By definition, each interval of depth n in the partition P_n gathers all numbers in $[0, 1] \setminus E$ whose representations, as sequences of labels, coincide until depth n.

[1] Of course, it would be possible to define (two) finite expansions for the elements in E, but we choose to concentrate just on $[0, 1] \setminus E$, the set corresponding to infinite expansions.

Remark 4.3 Examples of labeled sequences of partitions are given below. When each interval in P_n is divided into proper subintervals in P_{n+1}, the labels ρ_{n+1} usually produce digits that will be used to provide suitable expansions of the real numbers of the unit interval. This is for instance the case of the binary numeration (see Sect. 4.4). For other sequences of partitions, such as the Farey one in Sect. 4.6, the labeling ρ_n might seem to be less relevant at first view. However, we will see that it is connected to the coding of dynamical systems in the context of characteristic Sturmian words, as developed in Sect. 4.6.2.

4.3 Fibred systems

Often, a sequence of partitions, which allows for the definition of a numeration system as described in Sect. 4.2, can be defined in dynamical terms. A particularly relevant framework in this setting is the one of fibred systems as introduced in [42], and in [10, 15] for the notion of a number-theoretic fibred system.

In this paper, we say that the pair $([0, 1], T)$ is a *fibred system* if the transformation $T : [0, 1] \to [0, 1]$ is such that there exist a finite or countable set $\mathcal{A}$ and a topological partition $P = \{I_a\}_{a\in\mathcal{A}}$ of $[0, 1]$ for which the restriction T_a of T to I_a is injective and continuous, for each $a \in \mathcal{A}$.

We associate with a fibred system $([0, 1], T)$ a sequence of partitions in the usual dynamical way. Let $T^{-1}P$ denote the partition $\{T^{-1}I_a\}_{a\in\mathcal{A}}$. We define, for any $n \geq 1$, the partition P_n as the join partition $\bigvee_{i=0}^{n-1} T^{-j} P$ made up of the sets of the form $\bigcap_{j=0}^{n-1} T^{-j} I_{a_j}$, for all choices of $a_0, \ldots, a_{n-1} \in \mathcal{A}$, where each $I_{a_j} \in P$. For $n = 0$, we define $P_0 = \{(0, 1)\}$. Now, we define a sequence of labeling maps $\{\rho_n\}_{n\geq 1}$ as follows:

$$\rho_n \left(\bigcap_{j=0}^{n-1} T^{-j} I_{a_j} \right) := a_{n-1} \quad \text{for all choices of } a_0, \ldots, a_{n-1} \in \mathcal{A}.$$

Note that $\rho_n = \rho_1 \circ T^{n-1}$ for each n. If, moreover, $\mathcal{P} = \{P_n\}_{n\in\mathbb{N}}$ is log-balanced a.e. or in measure with respect to the Lebesgue measure and it is strictly self-refining, then $\mathcal{P}$ satisfies the conditions given in Sect. 4.2 and thus provides a numeration system $(\mathcal{P}, \rho)$. This numeration system associated with the map T together with the topological partition P (whose elements are indexed by the alphabet $\mathcal{A}$) is said to be, here, a *fibred numeration system*. The injectivity of the restriction T_a of T to I_a is consistent with the injectivity requirement of condition (H).

Fibred systems as above defined are particular cases of the more general fibred systems developed in [4]. For a fibred system $([0, 1], T)$ as above, the corresponding set X and the corresponding partition $\{X_a\}_{a\in\mathcal{A}}$ of X in the definition of fibred system given in Definition 2.3 of [4] are given by:

$$X = X(T, P) := \bigcap_{j=0}^{\infty} T^{-j} \bigcup_{a\in\mathcal{A}} I_a \quad \text{and} \quad X_a = X \cap I_a \text{ for each } a \in \mathcal{A}.$$

Clearly, $T(X) \subseteq X$ and the restriction T_a of T on X_a is injective. In our context, the set X equals the set $[0, 1] \setminus E$ where E is the set of endpoints of $\mathcal{P}$.

A fibred system $([0, 1], T)$ admits *an invariant and ergodic Borel probability measure* λ if, for every Borel set B, the following assertions hold: (i) $\lambda(T^{-1}(B)) = \lambda(B)$ and (ii) $T^{-1}(B) = B$ implies that B is a set of measure 0 or 1. We then denote it as $([0, 1], T, \lambda)$. According to [10, Definition 2.2], a fibred system is called a *number-theoretic fibred system* if T has an ergodic and invariant probability measure which is equivalent to the Lebesgue measure. A fibred system with an invariant and ergodic probability measure λ has then a well-defined entropy (see [40, Chap. 8]). The following theorem is a consequence of the Shannon–McMillan–Breiman Theorem (see [40, p. 134]).

Theorem 4.4 *Let* $([0, 1], T, \lambda)$ *be a fibred number-theoretic system (endowed with an ergodic invariant probability measure* λ *of the space* $[0, 1]$*). Let* $\mathcal{P} = \{P_n\}_{n\in\mathbb{N}}$ *be the sequence of partitions associated with this fibred system. If*

$$H_1(\mathcal{P}, \lambda) = -\sum_{I\in P_1} \lambda(I) \log \lambda(I) < \infty,$$

then for λ*-almost every* x*, the following limits exist and satisfy*

$$h = \lim_{n\to\infty} -\frac{\log \lambda(I_n(x))}{n} = -\lim_{n\to\infty} \frac{1}{n} \sum_{I\in P_n} \lambda(I) \log \lambda(I) .$$

In particular, the sequence of partitions $\mathcal{P}$ *is log-balanced a.e. with respect to* λ *and the map* $f : n \mapsto hn$ *is a corresponding weight function.*

Under the above assumption, we thus get, from a fibred number-theoretic system, a log-balanced sequence of partitions, as described in Sect. 4.2. By Proposition 3.4, if μ is another probability measure equivalent to λ, then $\mathcal{P}$ is also log-balanced a.e. for μ with f as weight function.

4.4 Fibred systems with positive entropy

Two classical instances of ergodic measure preserving fibred systems with positive entropy are the binary numeration system (in general, the beta-numeration) and the Gauss map producing continued fraction expansions. We describe them below.

4.4.1 The binary numeration system

The binary numeration system is the fibred number-theoretic system associated with the map $T : x \mapsto 2x \mod 1$ and the partition $P = \{(0, 1/2), (1/2, 1)\}$ with respect to the Lebesgue measure. Its entropy for the Lebesgue measure is $\log 2$. Its associated sequence of partitions is the *binary sequence of partitions* defined as $\mathcal{B} = \{B_n\}_{n\in\mathbb{N}}$, where

$$B_n = \left\{ \left(0, \frac{1}{2^n}\right), \left(\frac{1}{2^n}, \frac{2}{2^n}\right), \ldots, \left(\frac{2^n-1}{2^n}, 1\right) \right\}.$$

Hence, the set $E_n^{\mathcal{B}}$ of endpoints of the intervals in B_n consists of the points $a/2^n$ for $a = 0, 1, \ldots, 2^n$. The *binary sequence of partitions* $\mathcal{B}$ is a.e. log-balanced with weight function $f_{\mathcal{B}}(n) = (\log 2)n$ with respect to the Lebesgue measure.

4.4.2 The decimal numeration system

The decimal numeration system is the fibred number-theoretic system associated with the map $T : x \mapsto 10x \mod 1$ with the partition $P = \{(a/10, (a+1)/10)\}_{a=0,\ldots,9}$ with respect to the Lebesgue measure. We denote by $\mathcal{D} = \{D_n\}_{n\in\mathbb{N}}$ the *decimal sequence of partitions* associated with this fibred system. It admits as an a.e. weight function $f_{\mathcal{D}}(n) = (\log 10)n$ with respect to the Lebesgue measure.

4.4.3 Beta-numeration sequence of partitions

The *beta-numeration sequence of partitions* $\mathcal{BE}$ is the sequence of partitions associated with the the beta-numeration, i.e., the fibred system associated with the map $x \mapsto \beta x \mod 1$, where β is a given real number with $\beta > 1$. Its entropy for the Lebesgue measure is $\log \beta$. It is a.e. log-balanced with weight function $f_{\mathcal{BE}}(n) = (\log \beta)n$. This numeration has been introduced in [41]. For more on the length of the fundamental intervals, see e.g. [21]. When $\beta = 2$, one recovers the binary numeration and when $\beta = 10$, the decimal one.

4.4.4 Continued fraction sequence of partitions

The continued fraction expansions correspond to the fibred system associated with the Gauss map T_G such that

$$T_G(x) = \{1/x\} \quad \text{if } x \neq 0 \qquad \text{and} \qquad T_G(0) = 0,$$

where $\{x\}$ denotes the fractional part of x and the partition $\{(1/(a+1), 1/a)\}_{a\geq 1}$. The corresponding alphabet is the set of positive integers. The labeling map ρ_1 is given by $\rho_1(x) = \lfloor 1/x \rfloor$ and $\rho_n(x) = \rho_1(T_G^{n-1}(x))$. We denote by $\mathcal{CF}$ the sequence of partitions associated with this fibred system, called the *continued fraction sequence of partitions*.

We use the standard notation for continued fractions

$$x = [0; a_1(x), a_2(x), a_3(x), \ldots] = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \cdots}}}$$

with the last partial quotient larger than 1 in the case of rational numbers. As usual, the numerators and denominators of the n-th convergent $[0; a_1(x), a_2(x), \ldots, a_n(x)]$ are called continuants of the number x. They are denoted by $p_n = p_n(x)$ and $q_n = q_n(x)$. For any $n \geq 1$, the following recurrence relations hold

$$p_n = a_n p_{n-1} + p_{n-2},\ p_0 = 0,\ p_{-1} = 1,$$
$$q_n = a_n q_{n-1} + q_{n-2},\ q_0 = 1,\ q_{-1} = 0,$$

together with the equality

$$q_n p_{n-1} - p_n q_{n-1} = (-1)^n. \tag{5}$$

The fundamental interval $I_n^{\mathcal{CF}}(x)$ of depth n associated with x in the continued fraction expansion has as endpoints the fractions p_n/q_n and $(p_{n-1}+p_n)/(q_{n-1}+q_n)$. More precisely, by [30, p. 57], one has

$$I_n^{\mathcal{CF}}(x) = \begin{cases} \left(\dfrac{p_n}{q_n}, \dfrac{p_{n-1}+p_n}{q_{n-1}+q_n}\right) & \text{if } n \text{ is even} \\ \left(\dfrac{p_{n-1}+p_n}{q_{n-1}+q_n}, \dfrac{p_n}{q_n}\right) & \text{if } n \text{ is odd.} \end{cases} \tag{6}$$

In particular,

$$\left| I_n^{\mathcal{CF}}(x) \right| = \frac{1}{q_n(q_n + q_{n-1})}. \tag{7}$$

Let the *Gauss measure* be $d\mu = 1/((1+x)\log 2)\, dx$. Since T_G is an ergodic measure preserving transformation with respect to the Gauss measure (see, for instance, [40, p. 106]), the dynamical system $([0, 1], T_G, \mu)$ is a fibred number-theoretic system. Moreover, as the Gauss measure is absolutely continuous with respect to the Lebesgue measure, it can be shown, by relying on Theorem 4.4, that the entropy of the continued fraction sequence of partitions $\mathcal{CF}$ is $\pi^2/(6\log 2)$, with respect to both its invariant measure and the Lebesgue measure (see [28, Cor. 4.1.28] for a detailed proof). It follows that the sequence of partitions $\mathcal{CF}$ admits as an a.e. weight function

$$f_{\mathcal{CF}}(n) = \frac{\pi^2}{6\log 2} n \tag{8}$$

with respect to the Lebesgue measure (and the Gauss measure as well).

An alternative derivation of the value of the a.e. entropy of $\mathcal{CF}$ follows from Khinchin–Lévy's Theorem, which asserts that

$$\lim_{n\to\infty} \frac{\log q_n(x)}{n} = \frac{\pi^2}{12\log 2} \quad \text{a.e. } x. \tag{9}$$

We also recall the Borel–Bernstein Theorem (see [6, 9]).

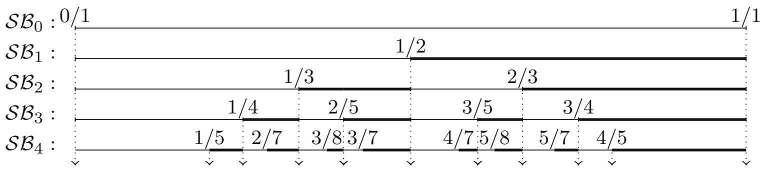

Fig. 1 The Stern–Brocot sequence of partitions

Theorem 4.5 (Borel–Berstein) *Let* $\{k_n\}_{n\in\mathbb{N}}$ *be a sequence of positive numbers. Then, the Lebesgue measure of* $\{x : a_n(x) \leq k_n$ *i.o.*$\}$ *is either* 0 *or* 1 *according as the series* $\sum_n 1/k_n$ *diverges or converges.*

4.5 A fibred numeration system with zero entropy: the Stern–Brocot system

The Stern–Brocot sequence of partitions, denoted as $\mathcal{SB}$, is the sequence of partitions associated with the Farey fibred system whose map is the Farey map

$$T_F = \begin{cases} \dfrac{1}{1-x} - 1 & = \dfrac{x}{1-x} \quad \text{if } x \in [0, 1/2] \\ \dfrac{1}{x} - 1 & = \dfrac{1-x}{x} \quad \text{if } x \in [1/2, 1] \end{cases}$$

and the partition $P = \{(0, 1/2), (1/2, 1)\}$. This sequence has been widely studied, see e.g. [37]. Even if the partition is nonuniform, considering logarithms of lengths allows the balance property in measure.

The standard construction of the sequence $\mathcal{SB} = \{SB_n\}_{n\in\mathbb{N}}$ is as follows (see Fig. 1 for an illustration): we start with $SB_0 = \{(0/1, 1/1)\}$ and, for each $n \in \mathbb{N}$, SB_{n+1} arises from SB_n by dividing each interval $(p/q, p'/q')$ of SB_{n-1} into two subintervals by its mediant $(p + p')/(q + q')$.

We denote by $E_n^{\mathcal{SB}}$ the set of endpoints of the intervals in SB_n.

Let $x \in [0, 1]$ be an irrational number and let $\{p_m/q_m\}$ stand for its sequence of convergents. We recall that the convergents satisfy the following mediant-construction [30, pp. 14–15]: if m is even, then

$$\frac{p_m}{q_m} \leq x \leq \frac{p_{m+1}}{q_{m+1}} = \frac{a_{m+1}p_m + p_{m-1}}{a_{m+1}q_m + q_{m-1}} \leq \cdots \leq \frac{p_{m-1} + p_m}{q_{m-1} + q_m} \leq \frac{p_{m-1}}{q_{m-1}}, \tag{10}$$

whereas if m is odd, then

$$\frac{p_{m-1}}{q_{m-1}} \leq \frac{p_{m-1} + p_m}{q_{m-1} + q_m} \leq \cdots \leq \frac{a_{m+1}p_m + p_{m-1}}{a_{m+1}q_m + q_{m-1}} = \frac{p_{m+1}}{q_{m+1}} \leq x \leq \frac{p_m}{q_m}. \tag{11}$$

The following result is well known (see [37, Lemma 1]).

Proposition 4.6 *Let $x \in [0,1]$ be an irrational number and let $n \in \mathbb{N}$. Then, the interval in SB_n containing x is*

$$I_n^{\mathcal{SB}}(x) = \begin{cases} \left(\dfrac{p_m}{q_m}, \dfrac{(r+1)p_m + p_{m-1}}{(r+1)q_m + q_{m-1}}\right) & \text{if } m \text{ is even} \\ \left(\dfrac{(r+1)p_m + p_{m-1}}{(r+1)q_m + q_{m-1}}, \dfrac{p_m}{q_m}\right) & \text{if } m \text{ is odd,} \end{cases}$$

where $m := m(x,n)$ and $r := r(x,n)$ are the unique integers such that

$$\sum_{i=1}^{m} a_i \le n < \sum_{i=1}^{m+1} a_i \quad \text{and} \quad r = n - \sum_{i=1}^{m} a_i.$$

As a consequence, one has

$$\left|I_n^{\mathcal{SB}}(x)\right| = \frac{1}{((r+1)q_m + q_{m-1})q_m}.$$

The next two results show that, with respect to the Lebesgue measure, the Stern–Brocot sequence of partitions is log-balanced in measure but not a.e.

Proposition 4.7 *The Stern–Brocot sequence of partitions admits as weight function in measure with respect to the Lebesgue measure*

$$f_{\mathcal{SB}}(n) := \frac{\pi^2}{6}\frac{n}{\log n} \quad \text{for each } n \ge 2.$$

Proof Let $x \in [0,1]$ be an irrational number and let $n \in \mathbb{N}$. Let m and r be as in Proposition 4.6, i.e., $\sum_{i=1}^{m} a_i \le n < \sum_{i=1}^{m+1} a_i$ and $r = n - \sum_{i=1}^{m} a_i$. Hence,

$$\frac{1}{2(r+1)q_m^2} \le \left|I_n^{\mathcal{SB}}(x)\right| \le \frac{1}{(r+1)q_m^2}.$$

Therefore, one has

$$-\frac{\log|I_n^{\mathcal{SB}}(x)|}{n/\log n} = \frac{\log(r+1)}{n/\log n} + 2\frac{\log q_m}{n/\log n} + O\left(\frac{\log n}{n}\right) \quad \text{as } n \to \infty. \tag{12}$$

Since $r \le n$, the first term in the above right-hand side tends to zero when $n \to \infty$ a.e. We write

$$\frac{\log q_m}{n/\log n} = \frac{\log q_m}{m}\,\frac{\log n}{\log m}\,\frac{m\log m}{n}. \tag{13}$$

Notice that, by (9), the first factor $\frac{\log q_m}{m}$ on the above right-hand side tends to $\frac{\pi^2}{12\log 2}$ as $m \to \infty$ a.e. Let us prove that

$$\lim_{n\to\infty}\frac{\log n}{\log m} = 1 \quad \text{a.e. } x. \tag{14}$$

(We recall that m depends on x and n.) By [17, Corollary 1], for a.e. x and large enough m (depending on x),

$$\sum_{i=1}^{m} a_i \le \frac{1+o(1)}{\log 2} m \log m + \max_{1\le i\le m} a_i .$$

Fix $\epsilon > 0$. By Borel–Bernstein Theorem (see Theorem 4.5), for a.e. x and for m large enough (depending on x), one has

$$a_m \le m(\log m)^{1+\epsilon}.$$

For a.e. x and n large enough (depending on x), this yields

$$\begin{aligned} m \le n < \sum_{i=1}^{m+1} a_i &\le \left(\frac{1+o(1)}{\log 2} \log(m+1) + (\log(m+1))^{1+\epsilon} \right) (m+1) \\ &= O(m(\log m)^{1+\epsilon}). \end{aligned}$$

Hence, $\log n = \log m + O(\log\log m)$ as $n \to \infty$, a.e. x. This proves (14).

In [31, p. 377], Khinchin proved that $\frac{\log 2}{m \log m} \sum_{i=1}^{m} a_i$ tends to 1 in measure as $m \to \infty$. Since, by definition, $\sum_{i=1}^{m} a_i \le n < \sum_{i=1}^{m+1} a_i$, this implies that

$$\lim_{n\to\infty} \frac{m \log m}{n} = \log 2 \quad \text{in measure.} \tag{15}$$

With Equation (13), the result follows. □

Proposition 4.8 *The Stern–Brocot sequence of partitions is not log-balanced a.e. with respect to the Lebesgue measure.*

Proof If the Stern–Brocot sequence of interval partitions $\mathcal{SB}$ were to admit an a.e. weight function f with respect to the Lebesgue measure, then, because of Proposition 4.7, $f(n)/f_{\mathcal{SB}}(n) \to 1$ as $n \to \infty$. Thus, it suffices to prove that $f_{\mathcal{SB}}$ is not an a.e. weight function for $\mathcal{SB}$.

Let $S_m = S_m(x, n) := \sum_{i=1}^{m} a_i$. Since $n \ge S_m$, by virtue of [38, Theorem 1] we have that

$$\liminf_{n\to\infty} \frac{m \log m}{n} \le \liminf_{n\to\infty} \frac{m \log m}{S_m} = 0 \quad \text{a.e. } x.$$

Notice however that (15) holds. As a consequence, $\frac{m \log m}{n}$ does not converge as $n \to \infty$ a.e. As the first two factors on the right-hand side of (13) converge as $n \to \infty$ a.e., $\log q_m/(n/\log n)$ does not and, consequently, neither does the left-hand side of (12). This proves that $f_{\mathcal{SB}}$ is not an a.e. weight function for $\mathcal{SB}$. □

The Stern–Brocot sequence of partitions has zero Shannon entropy with respect to the Lebesgue measure (see [8]). Moreover, the following also holds.

Proposition 4.9 *The Stern–Brocot sequence of partitions has zero entropy a.e. with respect to the Lebesgue measure.*

Proof Letting m be as in the proof of Proposition 4.7 and combining (12), (13), and (9), it follows that

$$-\frac{\log |I_n^{\mathcal{SB}}(x)|}{n} = O\left(\frac{m}{n}\right) \quad \text{as } n \to \infty, \text{ a.e. } x.$$

By [17, Corollary 1], for a.e. x, and for n large enough (depending on x), $n \geq \sum_{i=1}^{m} a_i \geq m \log m$. The proposition follows. □

4.6 The Farey sequence of partitions

In this section, we introduce another zero entropy sequence of partitions, namely the Farey sequence of partitions, and the closely related Sturmian words. The Farey sequence of partitions is a nonfibred example having zero entropy. The fact that it has indeed zero entropy follows from Proposition 3.6 because it has polynomially many intervals at each depth.

4.6.1 The Farey sequence of partitions $\mathcal{F}$

The *Farey sequence of partitions* $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ is defined as follows. Each of the partitions F_n is determined by the set $E_n^{\mathcal{F}}$ of endpoints of its intervals, i.e.,

$$E_n^{\mathcal{F}} := \left\{\frac{p}{q} :\ p, q \geq 1,\ \gcd(p, q) = 1 \text{ and } q \leq n+1\right\}.$$

The set of endpoints $E_n^{\mathcal{F}}$ thus corresponds to the *Farey sequence of order* $n+1$ which is the sequence of fractions h/k with $(h, k) = 1$ and $1 \leq h \leq k \leq n+1$, arranged in increasing order between 0 and 1, and studied e.g. in [25, 29].

Equivalently, $\mathcal{F}$ can be built recursively as follows. Let $F_0 = \{(0/1, 1/1)\}$ and, for each $n \in \mathbb{N}$, let F_{n+1} be the partition that arises from F_n by dividing each of the intervals $(p/q, p'/q')$ such that $q + q'$ is at most $n+2$ into two subintervals by its mediant $(p + p')/(q + q')$, while keeping all the other intervals of F_n unchanged.

The construction of the Farey sequence of partitions closely resembles the construction of the Stern–Brocot sequence of partitions given in Sect. 4.5. In fact, the only difference is that when constructing the sequence of partitions $\mathcal{SB}$ we divide every interval $(a/c, b/d)$ of SB_n into two subintervals, whereas when constructing $\mathcal{F}$ only those intervals $(p/q, p'/q')$ of F_n which satisfy $q + q' \leq n+2$ are divided. Compare Figs. 1 and 2.

By the definition of $E_n^{\mathcal{F}}$, it is clear that F_n consists of $O(n^2)$ intervals and thus Proposition 3.6 implies that $\mathcal{F}$ has zero entropy. Notice that, instead, each partition SB_n consists of 2^n intervals.

Let us recall some basic facts regarding the Farey sequence.

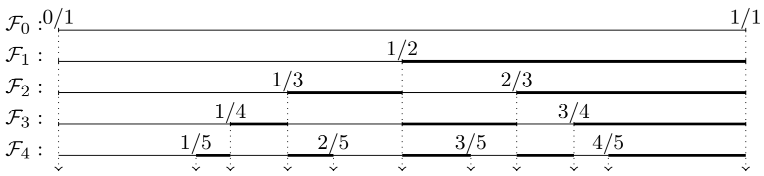

Fig. 2 The Farey sequence of partitions

Theorem 4.10 ([25, 26]) *Two irreducible fractions p/q and p'/q' in $[0, 1]$ are consecutive endpoints in $\mathcal{F}_n$ if and only if $q \le n+1$, $q' \le n+1$, $|p'q - pq'| = 1$ and $q + q' > n + 1$.*

The following result is probably folklore. However, given the lack of a suitable reference for it, we give a precise statement and a proof for the sake of completeness.

Proposition 4.11 *Let $x \in [0, 1]$ be an irrational and let $n \in \mathbb{N}$. Then, the interval in F_n containing x is*

$$I_n^{\mathcal{F}}(x) = \begin{cases} \left(\dfrac{p_m}{q_m}, \dfrac{(r+1)p_m + p_{m-1}}{(r+1)q_m + q_{m-1}}\right) & \text{if } m \text{ is even} \\ \left(\dfrac{(r+1)p_m + p_{m-1}}{(r+1)q_m + q_{m-1}}, \dfrac{p_m}{q_m}\right) & \text{if } m \text{ is odd,} \end{cases}$$

where $m := m(x, n)$ and $r := r(x, n)$ are the unique integers such that

$$(r+1)q_m + q_{m-1} \le n+1 < (r+2)q_m + q_{m-1}, \ m \ge 0, \ \text{and } 0 \le r < a_{m+1}. \quad (16)$$

As a consequence,

$$|I_n^{\mathcal{F}}(x)| = \frac{1}{((r+1)q_m + q_{m-1})q_m}.$$

Proof The uniqueness of m follows from the fact that (16) implies

$$q_m + q_{m-1} \le n+1 < (a_{m+1}+1)q_m + q_{m-1} = q_{m+1} + q_m.$$

The uniqueness of m in turn implies the uniqueness of r.

Notice that (10) or (11) (according to whether m is even or odd) implies that x belongs to the interval whose endpoints are the quotients p_m/q_m and $((r+1)p_m + p_{m-1})/((r+1)q_m + q_{m-1})$. As we are assuming that $q_m \le (r+1)q_m + q_{m-1} \le n+1$, both quotients are endpoints of F_n. Moreover, Theorem 4.10 implies that these quotients are consecutive endpoints of F_n because $q_m + ((r+1)q_m + q_{m-1}) = (r+2)q_m + q_{m-1} > n+1$ and

$$((r+1)p_m + p_{m-1})q_m - ((r+1)q_m + q_{m-1})p_m = p_{m-1}q_m - q_{m-1}p_m = 1$$

by (5). □

Remark 4.12 Notice, by comparing Propositions 4.6 and 4.11, the intervals in the Stern–Brocot $\mathcal{SB}$ and in the Farey $\mathcal{F}$ sequences of partitions that contain a given x have the same form, however they might occur at different indices for their depth.

Proposition 4.13 *The Farey sequence of partitions admits as a.e. weight function with respect to the Lebesgue measure*

$$f_{\mathcal{F}}(n) = 2\log n \quad \textit{for each } n \geq 1.$$

Proof Let $x \in [0, 1]$ be an irrational number. Let $m \geq 0$ as in Proposition 4.11 (i.e., $(r+1)q_m + q_{m-1} \leq n+1 < (r+2)q_m + q_{m-1}$ where $0 \leq r < a_{m+1}$). Notice that $q_m \leq (r+1)q_m + q_{m-1} \leq n+1$. Notice also that $n+1 < (r+2)q_m + q_{m-1} \leq (r+3)q_m$. Similarly, $n+1 < (r+2)q_m + q_{m-1} \leq 2((r+1)q_m + q_{m-1})$. In this way, we proved that

$$\frac{1}{(n+1)^2} \leq \left|I_n^{\mathcal{F}}(x)\right| \leq \frac{2(r+3)}{(n+1)^2}.$$

It is well-known that $m = O(\log n)$ as $n \to \infty$ (since $2^{(m-1)/2} \leq q_m \leq n$). Hence, since Borel–Bernstein Theorem (Theorem 4.5) ensures that $a_m = O(m(\log m)^2)$ as $m \to \infty$, for a.e., it follows that $r < a_{m+1} = O(\log n (\log\log n)^2)$ a.e. As a consequence,

$$-\log|I_n^{\mathcal{F}}(x)| = 2\log(n+1) + O(\log\log n) \quad \text{as } n \to \infty, \quad \text{a.e. } x,$$

where the hidden constants in the O-term may depend on x. The result follows. □

4.6.2 The Farey sequence of partitions and Sturmian words

The Farey sequence of partitions has a particular combinatorial meaning in symbolic dynamics by producing prefixes of so-called characteristic Sturmian words. Indeed, given an irrational real number $\alpha \in [0, 1]$ consider the Kronecker-Weyl sequence $\{n\alpha\}_{n\geq 1}$ (here $\{x\}$ is the fractional part of x). Sturmian words are obtained as binary codings of Kronecker-Weyl sequences, thus providing a numeration system as introduced in Sect. 4.2 for irrational numbers with digits in $\{0, 1\}$. We describe it below.

Let α be an irrational number in $[0, 1]$. Consider the two intervals $(0, 1-\alpha)$ and $(1-\alpha, 1)$. We define the sequence $S(\alpha) := \{s_n(\alpha)\}_{n\geq 1}$ in $\{0, 1\}^{\mathbb{N}}$ associated with α as follows:

$$s_n(\alpha) = \begin{cases} 0 & \text{if } \{n\alpha\} \in (0, 1-\alpha) \\ 1 & \text{if } \{n\alpha\} \in (1-\alpha, 1). \end{cases}$$

Since α is assumed to be irrational, observe that the sequence $\{n\alpha\}$ never takes the value 0, nor $1-\alpha$. We will use the notation $S(\alpha)_{[1,n]}$ for the prefix $s_1 \cdots s_n$ of length n of the sequence $S(\alpha)$ (considered as an infinite word over the alphabet $\{0, 1\}$).

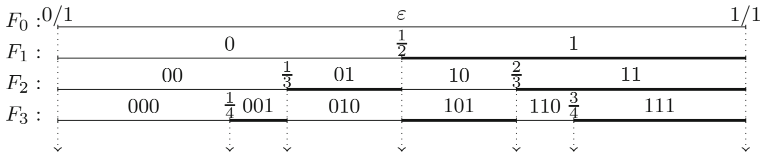

Fig. 3 The labeling of the Farey partition F_n for $n \le 3$

The sequence $S(\alpha)$ is a so-called *characteristic Sturmian word* (see e.g. [35, Chapter 2]). Sturmian words are among the most studied words in word combinatorics and symbolic dynamics.

By [7, Lemma 5], irrational numbers α that belong to a common interval of the partition F_n have the same prefix of length n for the characteristic Sturmian word $s_n(\alpha)$, while these prefixes differ if they belong to two distinct intervals of F_n. Hence, with each interval in the Farey partition F_n of order n is associated the prefix of length n of some characteristic Sturmain word $S(\alpha)$.

More precisely, according to [7, 36], let $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ be the endpoints of an interval in F_n with $\frac{p_1}{q_1} \neq 0$, $\frac{p_2}{q_2} \neq 1$, $\frac{p_1}{q_1} < \frac{p_2}{q_2}$ and $q_1 + q_2 \le n + 2$. Let α be an irrational number in $[\frac{p_1}{q_1}, \frac{p_2}{q_2}]$. If α belongs to $[\frac{p_1}{q_1}, \frac{p_1+p_2}{q_1+q_2}]$, then $S(\alpha)_{[1,n+1]} = S(\alpha)_{[1,n]}0$, and if $\alpha \in [\frac{p_1+p_2}{q_1+q_2}, \frac{p_2}{q_2}]$, then $S(\alpha)_{[1,n+1]} = S(\alpha)_{[1,n]}1$. Moreover, for all irrationals α in $[\frac{p_1}{q_1}, \frac{p_2}{q_2}]$, $S(\alpha)]_{[1,n+1]}$ is a palindrome if and only if $q_1 + q_2 = n + 2$. This thus allows the definition of a labeling function $\rho_n : F_n \to \{0, 1\}$ such as introduced in Sect. 4.2 that maps each interval of F_n to the last letter s_{n+1} of the prefix $S(\alpha)_{[1,n+1]}$ of the characteristic Sturmian word $S(\alpha)$. The labeling function $\rho = \{\rho_n\}_{n \ge 1}$ of the Farey sequence of partitions thus works as follows. We begin with the coding of the unique interval in F_0 as the empty word. Then, there are two cases: when the mediant of an interval of F_n can be added as an endpoint of F_{n+1}, then this interval is subdivided and the coding of the two subintervals is made in the lexicographic order by adding a letter in $\{0, 1\}$. When the mediant cannot be added, the interval is not subdivided, and the coding is extended in a unique way by adding the next letter in its palindromic completion. This is illustrated in Fig. 3.

Remark 4.14 Consider the case where we consider two identical sequences of partitions $\mathcal{P} = \mathcal{P}^1 = \mathcal{P}^2$ in Definition 2.5, with $\mathcal{P}$ log-balanced (a.e. or in measure). One can have $L_n(x, \mathcal{P}, \mathcal{P}) > n$ if $I_n(x) = I_{n+1}(x)$. This is for instance the case for the Farey partition. The construction of the coding using the palindromic completion which shows that one might have more digits in an interval of F_n than the $n + 1$ letters of a prefix. For instance, $(1/(n + 1), 1/n) \in F_n$ and $L_n(x, \mathcal{F}, \mathcal{F}) = 2n$ for every $x \in (1/(n + 1), 1/n)$.

4.7 The three-distance sequence of partitions

The *three-distance sequence of partitions* $3\mathcal{D}(\alpha)$ is defined in terms of the corresponding endpoints of each partition $3\mathcal{D}_n(\alpha)$. The set $E_n^{3\mathcal{D}(\alpha)}$ of endpoints of $3\mathcal{D}_n(\alpha)$

is given by $\{1\} \cup \{(i\alpha) \bmod 1 \,:\, 0 \le i \le n\}$. We remark that, when α is irrational, these determine exactly $n+1$ intervals.

4.7.1 Almost everywhere log-balancedness

It turns out, somewhat surprisingly, see [43–45] or the survey [3], that the lengths of the intervals in $3\mathcal{D}_n(\alpha)$ can only take at most three different values. This classical result is known as the *three-distance theorem*, which we cite below. We make use of this theorem to derive the weight function of the three-distance partition for almost every α. See [12] for a probabilistic study of the lengths in the case where there are two distances, [39] for a study of the distribution of the lengths when averaging over α, and see also [48].

Theorem 4.15 (The three-distance theorem) *Let $\alpha \in (0,1)$ an irrational number and n be a positive integer. The points $\{1\} \cup \{(i\alpha) \bmod 1 \,:\, 0 \le i \le n\}$ partition the unit interval $[0,1]$ into $n+1$ intervals, the lengths of which take at most three values, one being the sum of the other two.*

More precisely, let $\{p_k/q_k\}_{k\in\mathbb{N}}$ and $\{a_k\}_{k\in\mathbb{N}}$ be the sequences of the convergents and partial quotients associated with α in its continued fraction expansion. There exist unique integers k, m, and r such that

$$n = mq_k + q_{k-1} + r,\ k \ge 0,\ 1 \le m \le a_{k+1},\ \text{and } 0 \le r < q_k.$$

Let $\eta_k = (-1)^k(q_k\alpha - p_k)$ for every $k \in \mathbb{N}$. Then, the unit interval is divided by the points $\{\alpha\}, \{2\alpha\}, \ldots, \{n\alpha\}$ into $n+1$ intervals which satisfy that:

- *$n+1-q_k$ of them have length η_k (which is the smallest of the three lengths),*
- *$r+1$ of them have length $\eta_{k-1} - m\eta_k$, and*
- *$q_k - (r+1)$ of them have length $\eta_{k-1} - (m-1)\eta_k$ (which is the largest of the three lengths).*

There is an interesting connection between this theorem and frequencies of words that occur in a characteristic Sturmian word (see Sect. 4.6.2). Indeed, the lengths of intervals in Theorem 4.15 coincide with the frequencies of factors of length $n+1$ [7]. We will repeatedly use the following result.

Theorem 4.16 ([30, Theorems 9 and 13]) *Let $\alpha \in (0,1)$ an irrational and $\{p_k/q_k\}_{k\in\mathbb{N}}$ its sequence of convergents. If $k \in \mathbb{N}$ and $\eta_k = (-1)^k(q_k\alpha - p_k)$, then*

$$\frac{1}{q_{k+1}+q_k} < \eta_k < \frac{1}{q_{k+1}}.$$

4.7.2 Weight function for most three-distance sequences of partitions $3\mathcal{D}(\alpha)$

We will prove that, for almost every α, the three-distance sequence of partitions $3\mathcal{D}(\alpha)$ is a.e. log-balanced as a consequence of the following lemma.

Lemma 4.17 *For almost every α in $(0, 1)$, the three-distance sequence of partitions $3\mathcal{D}(\alpha)$ satisfies*

$$-\log|I_n^{3\mathcal{D}(\alpha)}(x)| = \log n + O(\log\log n) \text{ as } n \to \infty\,, \quad \text{for every } x \notin E^{3\mathcal{D}(\alpha)},$$

where the hidden constant in the O-term might depend on α, but can be chosen so that the result holds for every $x \notin E^{3\mathcal{D}(\alpha)}$. Moreover, the limit $-\log|I_n^{3\mathcal{D}(\alpha)}(x)|/\log n \to 1$ as $n \to \infty$ holds for any irrational $\alpha \in (0, 1)$ such that $\log a_k(\alpha) = o(k)$ as $k \to \infty$.

Proof Let α be an irrational in $(0, 1)$. We keep the notation introduced in Theorem 4.15. In particular, $\eta_k := |q_k(\alpha)\cdot\alpha - p_k(\alpha)|$ for each $k \in \mathbb{N}$.

Let n be a positive integer and choose $k \in \mathbb{N}$ so that

$$q_k + q_{k-1} \le n < q_{k+1} + q_k.$$

(we recall $q_0 = 1$ and $q_{-1} = 0$). Let $x \in [0, 1] \setminus E^{3\mathcal{D}(\alpha)}$. By Theorem 4.15, $\{\eta_k\}_{k\in\mathbb{N}}$ is decreasing, $\eta_{k+1} = -a_{k+1}\eta_k + \eta_{k-1}$, and

$$\eta_k \le |I_n^{3\mathcal{D}(\alpha)}(x)| \le \eta_{k-1}.$$

By Theorem 4.16, $\log q_{k+1} < -\log\eta_k < \log(q_{k+1} + q_k)$ and we obtain

$$\log q_k < -\log\eta_{k-1} \le -\log|I_n^{3\mathcal{D}(\alpha)}(x)| \le -\log\eta_k < \log(q_{k+1} + q_k).$$

Since $q_{k+1} + q_k = (a_{k+1} + 1)q_k + q_{k-1} < (a_{k+1} + 2)q_k \le (a_{k+1} + 2)n$,

$$-\log|I_n^{3\mathcal{D}(\alpha)}(x)| \le \log(q_{k+1} + q_k) < \log n + \log(a_{k+1} + 2).$$

But we also obtain $q_k > (q_{k+1} + q_k)/(a_{k+1} + 2) > n/(a_{k+1} + 2)$, which implies

$$-\log|I_n^{3\mathcal{D}(\alpha)}(x)| \ge \log q_k > \log n - \log(a_{k+1} + 2).$$

The Borel–Bernstein Theorem (Theorem 4.5) implies that, for a.e. α, $a_{k+1} \le k^2$ for k large enough. Moreover, $k = O(\log n)$ as $n \to \infty$ for every irrational $\alpha \in (0, 1)$, as $2^{(k-1)/2} \le q_k(\alpha) \le n$. Therefore, $\log(a_{k+1} + 2) = O(\log\log n)$ as $n \to \infty$, for a.e. α. This proves the first assertion of the lemma.

The second assertion of the lemma follows from the fact that if $\log(a_k) = o(k)$ as $k \to \infty$, then $\log(a_{k+1} + 2) = o(\log n)$ as $n \to \infty$. □

We have the following consequence of Lemma 4.17.

Proposition 4.18 *For almost every α in $(0, 1)$, the three-distance sequence of partitions $3\mathcal{D}(\alpha)$ is a.e. log-balanced with respect to the Lebesgue measure with weight function*

$$f_{3\mathcal{D}(\alpha)}(n) = \log n \quad \text{for each } n \ge 1.$$

More specifically, it holds for any irrational $\alpha \in (0, 1)$ whose continued fraction expansion satisfies $\log a_k(\alpha) = o(k)$ as $k \to \infty$.

Remark 4.19 The above result holds, for instance, for $\alpha = \phi^{-2}$, where ϕ denotes the golden ratio, and for $\alpha = e - 2 = [0; \overline{1, 2k, 1}_{k=1}^{\infty}]$ (see e.g., [14]).

4.7.3 Sequences of partitions that are not log-balanced

The next proposition will provide an uncountable family of numbers α for which the corresponding sequence of partitions $3\mathcal{D}(\alpha)$ is not log-balanced.

Proposition 4.20 *Fix any real number $s > 0$. We define $\alpha = \alpha(s) \in (0, 1)$ as the number whose continued fraction expansion satisfies the following recurrence relation*

$$a_1(\alpha) = 1 \quad \text{and} \quad a_{k+1}(\alpha) = \lceil q_k^s(\alpha) \rceil \quad \text{for each } k \geq 1.$$

Then, the corresponding $3\mathcal{D}(\alpha)$ sequence of partitions is not log-balanced in measure (thus, neither a.e. log-balanced) with respect to the Lebesgue measure.

Proof We will prove that

$$\frac{-\log |I_n^{3\mathcal{D}(\alpha)}(x)|}{\log n}$$

does not converge in measure with respect to the Lebesgue measure as $n \to \infty$. For this purpose, let us define $\{n_k\}_{k\in\mathbb{N}}$ as follows: for each $k \in \mathbb{N}$,

$$n_k = m_k q_k + q_{k-1} + r_k, \quad \text{with} \quad m_k = \left\lceil \frac{a_{k+1}}{2} \right\rceil \quad \text{and} \quad r_k = q_k - 1.$$

According to the three-distance theorem, there are only two different lengths among the intervals in $3\mathcal{D}_{n_k}(\alpha)$. More precisely, these $n_k + 1$ intervals can be partitioned into two types:

- $n_k + 1 - q_k$ intervals of length η_k and
- $r_k + 1 = q_k$ intervals of length $\Delta_k := \eta_{k-1} - m_k \eta_k$.

On the basis of this remark, we will prove that for our choice of $\{n_k\}_{k\in\mathbb{N}}$ and for each $x \in [0, 1] \setminus E^{3\mathcal{D}(\alpha)}$, one has:

$$\frac{-\log |I_{n_k}^{3\mathcal{D}(\alpha)}(x)|}{\log n_k} = g(x, k) + O\left(\frac{1}{k}\right) \quad \text{as } k \to \infty, \tag{17}$$

where

$$g(x, k) := \begin{cases} 1 & \text{if } |I_{n_k}^{3\mathcal{D}(\alpha)}(x)| = \eta_k, \\ \dfrac{1}{s+1} & \text{if } |I_{n_k}^{3\mathcal{D}(\alpha)}(x)| = \Delta_k, \end{cases}$$

and the constant hidden in the O-term does not depend on x.

In order to estimate the three quantities n_k, η_k, and Δ_k by a convenient power of q_k, we will bound m_k as follows:

$$\frac{1}{2}q_k^s \leq \frac{a_{k+1}}{2} \leq m_k < \frac{a_{k+1}}{2} + 1 < \frac{1}{2}(q_k^s + 1) + 1 = \frac{1}{2}q_k^s\left(1 + \frac{3}{q_k^s}\right). \tag{18}$$

Hence, each depth n_k satisfies

$$\frac{1}{2}q_k^{s+1} \leq n_k < m_k q_k + 2q_k = \frac{1}{2}q_k^{s+1}\left(1 + \frac{7}{q_k^s}\right). \tag{19}$$

Now, we bound the distance η_k. Since $q_k^s \leq a_{k+1} < q_k^s + 1$, the continuant q_{k+1} satisfies

$$q_{k+1} < (q_k^s + 1)q_k + q_{k-1} < (q_k^s + 1)q_k + q_k = q_k^{s+1}\left(1 + \frac{2}{q_k^s}\right) \quad \text{and}$$
$$q_{k+1} > q_k^{s+1}.$$

Thus, by Theorem 4.16, each length η_k satisfies

$$\frac{q_k^{-(s+1)}}{1 + 3/q_k^s} \leq \frac{1}{q_{k+1} + q_k} < \eta_k < \frac{1}{q_{k+1}} \leq q_k^{-(s+1)}. \tag{20}$$

Inequalities (19) and (20) prove that $\log n_k = (s+1)\log q_k + O(1)$ and also $-\log \eta_k = (s+1)\log q_k + O(1)$ as $k \to \infty$. Since $q_k = q_k(\alpha)$ has, at least, exponential growth with respect to k, then

$$\frac{-\log \eta_k}{\log n_k} = 1 + O\left(\frac{1}{k}\right) \quad \text{as } k \to \infty. \tag{21}$$

In order to bound the length Δ_k, we remark that the above upper bound on η_k and the upper bound on m_k given in (18) yield

$$m_k \eta_k < \frac{q_k^{-1}}{2}\left(1 + \frac{3}{q_k^s}\right). \tag{22}$$

By construction, the sequence of partial quotients $\{a_k\}$ tends to infinity as k does. Hence, it is possible to choose a large enough $k_0 \in \mathbb{N}$ so that $q_{k-1}/q_k \leq 1/3$. Without loss of generality, we assume k_0 is large enough so that $3/q_k^s \leq 1/4$ for any $k \geq k_0$. For such a k, it follows by Theorem 4.16 that

$$\Delta_k = \eta_{k-1} - m_k \eta_k < q_k^{-1} \quad \text{and}$$
$$\Delta_k \geq q_k^{-1}\left(\frac{1}{1 + \frac{q_{k-1}}{q_k}}\right) - \frac{q_k^{-1}}{2}\left(1 + \frac{3}{q_k^s}\right) \geq \frac{1}{8}q_k^{-1}.$$

These inequalities prove that $-\log \Delta_k = \log q_k + O(1)$ but, according to (19), $-\log n_k = (s+1)\log q_k + O(1)$, as $k \to \infty$. Thus, as the continuants q_k grow at least exponentially with k, this yields

$$\frac{-\log \Delta_k}{\log n_k} = \frac{1}{s+1} + O\left(\frac{1}{k}\right) \quad \text{as } k \to \infty. \tag{23}$$

Equation (17) now follows from (21) and (23). Notice that (17) implies that there are three possibilities for the sequence $-\log|I_{n_k}^{3\mathcal{D}(\alpha)}(x)|/\log n_k$ depending on $x \in [0,1] \setminus E^{3\mathcal{D}(\alpha)}$: (i) it has a limit equal to 1; (ii) it has a limit equal to $1/(s+1)$; or (iii) it has no limit. To end the proof of this proposition, we now show that there exists $\epsilon > 0$ so that the measures of the sets

$$F_{k,\epsilon} = \left\{ x \in [0,1] \setminus E^{3\mathcal{D}(\alpha)} \,:\, \left| \frac{-\log|I_{n_k}^{3\mathcal{D}(\alpha)}(x)|}{\log n_k} - 1 \right| < \varepsilon \right\}$$

belong to $\left[\frac{1}{4}, \frac{3}{4}\right]$ for k large enough. Consider any ϵ with $0 < \epsilon < 1/(2(s+1))$. Because of (17), there is a $k_1 \in \mathbb{N}$ so that, for each $k \geq k_1$, the set $F_{k,\epsilon}$ coincides with the set G_k given by

$$G_k = \left\{ x \in [0,1] \setminus E^{3\mathcal{D}(\alpha)} \,:\, |I_{n_k}^{3\mathcal{D}(\alpha)}(x)| = \eta_k \right\}.$$

The number of intervals of length η_k is $n_k + 1 - q_k$. Thus, one has

$$|G_k| = (n_k + 1 - q_k)\eta_k = (m_k q_k + q_{k-1})\eta_k.$$

The bounds on $m_k\eta_k$ and η_k in (22) and (20) yield, for k large enough so that $5/q_k^s \leq 1/2$,

$$|G_k| \leq (m_k+1)\eta_k q_k \leq \frac{1}{2} q_k^{-1}\left(1 + \frac{5}{q_k^s}\right) q_k \leq \frac{3}{4} \quad \text{and}$$
$$|G_k| \geq m_k q_k \eta_k \geq \frac{1}{2} q_k^{s+1} \left(\frac{q_k^{-s-1}}{1 + 3/q_k^s}\right) \geq \frac{1}{4}.$$

Finally, we have proved that, for each ϵ with $0 < \varepsilon < 1/(2(s+1))$ and sufficiently large k, the following holds $\left|F_{k,\epsilon}\right| = |G_k| \in [1/4, 3/4]$. Then, the sequence $-\log|I_{n_k}^{3\mathcal{D}(\alpha)}(x)|/\log n_k$ cannot converge in measure as $k \to \infty$, and thus the proof of the proposition is complete. □

5 On the Farey and continued fraction sequences of partitions: an explicit case

In this section, we look at the relationship between the Farey and the continued fraction sequences of partitions. Both cases are of particular interest as the Lochs' indexes in both conversion directions can be computed explicitly, as demonstrated with Propositions 5.3 and 5.5, which moreover provide direct proofs of the Lochs-type theorems for these cases. We also show stronger results, when compared to our general theorems, including a probabilistic version of the error term in the case of the Lochs' indexes from continued fractions to Farey.

More precisely, for the probabilistic error term when going from continued fractions to Farey, namely Theorem 5.4, we apply a strong result regarding the asymptotic distribution of the logarithm of the continuants $\log q_n$ as $n \to \infty$. This result is a refinement over the classical Khinchin–Lévy's Theorem (see (9)), which tells us that $(1/n)\log q_n \to \pi^2/(12\log 2)$ as $n \to \infty$. The refinement comes in the form of an extra term, of a probabilistic nature: the random variable $\log q_n$ is asymptotically Gaussian (see, e.g., [24, Thm. 1] and [27]), as recalled below.

Theorem 5.1 *Let $h = \pi^2/(6\log 2)$ be the entropy of the continued fraction sequence of partitions. Then, there exists a positive constant B so that the sequence*

$$Z_n(x) = \frac{\log q_n(x) - \frac{h}{2}n}{B\sqrt{n}} \tag{24}$$

is asymptotically Gaussian, i.e., for every $t \in \mathbb{R}$,

$$|\{x \in [0,1] : Z_n(x) < t\}| = \frac{1}{\sqrt{2\pi}} \int_0^t e^{-w^2/2} dw + O\Big(\frac{1}{\sqrt{n}}\Big)$$

as $n \to \infty$. (We recall that $|\cdot|$ stands for the Lebesgue measure.)

5.1 From continued fractions to the Farey sequence of partitions

We begin by describing the Lochs' indexes for the conversion from the continued fraction sequence of partitions to the Farey one.

Lemma 5.2 *For each irrational $x \in (0,1)$ and each $n \in \mathbb{N}$, one has*

$$I_n^{\mathcal{CF}}(x) = I_k^{\mathcal{F}}(x), \quad \textit{where } k = q_n + q_{n-1} - 1. \tag{25}$$

Proof On the one hand, the interval $I_n^{\mathcal{CF}}(x)$ of the continued fraction sequence of partitions is simply, up to reordering the endpoints depending on the parity of n,

$$I_n^{\mathcal{CF}}(x) = \left(\frac{p_n}{q_n}, \frac{p_n + p_{n-1}}{q_n + q_{n-1}}\right). \tag{26}$$

On the other hand, by Proposition 4.11, the Farey interval of depth k containing x, up to reordering of the endpoints depending on the parity of m, is given by

$$I_k^{\mathcal{F}}(x) = \left(\frac{p_m}{q_m}, \frac{(r+1)p_m + p_{m-1}}{(r+1)q_m + q_{m-1}}\right), \tag{27}$$

where m and r are the only integers such that

$$(r+1)q_m + q_{m-1} \leq k+1 < (r+2)q_m + q_{m-1}, \; m \geq 0, \; \text{and } 0 \leq r < a_{m+1}.$$

Notice that if $k = q_n + q_{n-1}$, then $r = 0$ and $m = n$, which shows that $I_k^{\mathcal{F}}(x) = I_n^{\mathcal{CF}}(x)$. □

Proposition 5.3 *For any irrational number $x \in (0,1)$ and any integer $n \geq 1$, one has*

$$L_n(x, \mathcal{CF}, \mathcal{F}) = 2q_n(x) + q_{n-1}(x) - 2.$$

Proof By Lemma 5.2, $I_n^{\mathcal{CF}}(x) = I_k^{\mathcal{F}}(x)$, where $k = q_n + q_{n-1} - 1$. This interval will split into two subintervals in the Farey sequence of partitions for the first time at depth $2q_n + q_{n-1} - 1$ (for this is the denominator of the mediant of its endpoints minus 1). Hence, the proposition follows. □

Theorem 5.1, together with the above result, yields the following theorem which gives a Lochs-type theorem for the conversion from the continued fraction sequence of partitions to the Farey one with an error term satisfying an asymptotically Gaussian law.

Theorem 5.4 *If Z_n and B are as in Theorem 5.1 and x is an irrational in $(0,1)$, then*

$$\frac{\log L_n(x, \mathcal{CF}, \mathcal{F})}{n} = \frac{\pi^2}{12\log 2} + \frac{B}{\sqrt{n}} Z_n(x) + O\left(\frac{1}{n}\right), \quad \textit{as } n \to \infty, \tag{28}$$

with Z_n asymptotically Gaussian. In particular,

$$\lim_{n\to\infty} \frac{\log L_n(x, \mathcal{CF}, \mathcal{F})}{n} = \frac{\pi^2}{12\log 2} \quad \textit{a.e.},$$

with respect to the Lebesgue measure.

Proof Let $x \in (0,1)$ an irrational. Observe that, by Proposition 5.3 and each positive integer n,

$$\log L_n(x, \mathcal{CF}, \mathcal{F}) = \log(2q_n + q_{n-1} - 2) = \log q_n + \log\left(2 + \tfrac{q_{n-1}-2}{q_n}\right).$$

As the term $\log\left(2 + \tfrac{q_{n-1}-2}{q_n}\right)$ is upper bounded by $\log 3$, then

$$\frac{\log L_n(x, \mathcal{CF}, \mathcal{F})}{n} = \frac{\log q_n}{n} + O\left(\frac{1}{n}\right) = \frac{\pi^2}{12\log 2} + \frac{B}{\sqrt{n}} Z_n(x) + O\left(\frac{1}{n}\right)$$

as $n \to \infty$ by Theorem 5.1. □

5.2 Coming back: from Farey to continued fractions

In the case of the conversion from the Farey sequence of partitions to the continued fractions one, the Lochs-type result we obtain comes directly from the expression of the Lochs' index from Proposition 5.6 below. This is an almost everywhere result that is not covered by our more generally applicable Theorem 6.8. Indeed, the difficulty in applying Theorem 6.8 in this case stems from the fact that assertion (i) does not hold; this is because the weight function f_1 of the Farey sequence of partitions is slowly-increasing. Nevertheless, our general result in measure (Theorem 6.14) does apply in this case, which is enough to ensure that the limit (30) holds but only in measure.

Proposition 5.5 *Let $x \in (0,1)$ be an irrational number and let n be a positive integer. If $k = k(x,n)$ is the only positive integer such that*

$$q_k(x) + q_{k-1}(x) \leq n+1 < q_{k+1}(x) + q_k(x)\,, \tag{29}$$

then

$$L_n(x, \mathcal{F}, \mathcal{CF}) = k.$$

Proof By Lemma 5.2, $I_k^{\mathcal{CF}}(x) = I_{n'}^{\mathcal{F}}(x)$ and $I_{k+1}^{\mathcal{CF}}(x) = I_{n''}^{\mathcal{F}}(x)$ where $n' = q_k + q_{k-1} - 1$ and $n'' = q_{k+1} + q_k - 1$. By assumption, $n' \leq n < n''$. Thus,

$$I_{k+1}^{\mathcal{CF}}(x) = I_{n''}^{\mathcal{F}}(x) \subseteq I_n^{\mathcal{F}}(x) \subseteq I_{n'}^{\mathcal{F}}(x) = I_k^{\mathcal{CF}}(x).$$

Notice that one of the endpoints of $I_{k+1}^{\mathcal{CF}}(x)$ is $\frac{p_{k+1}+p_k}{q_{k+1}+q_k}$, which does not belong to F_n (because $q_{k+1} + q_k > n+1$). Hence, $I_{k+1}^{\mathcal{CF}}(x) \subsetneq I_n^{\mathcal{F}}(x)$ and, in particular, $I_n^{\mathcal{F}}(x) \nsubseteq I_{k+1}^{\mathcal{CF}}(x)$. We conclude that k is the largest integer k' such that $I_n^{\mathcal{F}}(x) \subseteq I_{k'}^{\mathcal{CF}}(x)$, i.e., $k = L_n(x, \mathcal{F}, \mathcal{CF})$. □

Proposition 5.6 *The following holds with respect to the Lebesgue measure:*

$$\lim_{n\to\infty} \frac{L_n(x; \mathcal{F}, \mathcal{CF})}{\log n} = \frac{12 \log 2}{\pi^2} \quad a.e. \tag{30}$$

Proof Let $x \in (0,1)$ an irrational and n be a positive integer. By Proposition 5.5, if $k = k(x,n)$ is the only positive integer satisfying (29), then $L_n(x, \mathcal{F}, \mathcal{CF}) = k(x,n)$ and the following inequalities hold

$$2q_{k(x,n)-1}(x) \leq n \leq 2q_{k(x,n)+1}(x).$$

As $k(x,n) \to \infty$ as $n \to \infty$, Khinchin–Lévy's Theorem (see (9)) yields $\log q_{k(x,n)+1} / \log q_{k(x,n)-1} \to 1$ as $n \to \infty$ a.e. Hence, $\log n / \log q_{k(x,n)} \to 1$ as

$n \to \infty$ a.e. Therefore, again by Khinchin–Lévy's Theorem, we have that

$$\lim_{n\to\infty} \frac{L_n(x, \mathcal{F}, \mathcal{CF})}{\log n} = \lim_{n\to\infty} \frac{k(x,n)}{\log n} = \lim_{n\to\infty} \frac{k(x,n)}{\log q_{k(x,n)}} = \frac{12\log 2}{\pi^2} \quad \text{a.e.}$$

□

It would be interesting to characterize

$$\frac{L_n(x, \mathcal{F}, \mathcal{CF}) - \frac{\pi^2}{12\log 2}\log n}{\sqrt{\log n}}$$

in law. The objective would be to obtain an analog to Theorem 5.4 but going from Farey to continued fractions.

6 Lochs-type theorems for log-balanced sequences of partitions

In this section, we give the proofs of our general Lochs-type theorems: Theorems 6.8 and 6.14. The structure of this section is as follows. In Sect. 6.1, we introduce some auxiliary notions which are useful for the proofs throughout this section. In Sects. 6.2 and 6.3, we prove our main result a.e. and in measure, respectively.

6.1 Preliminaries

We follow here the notion of ϵ-goodness introduced in [15].

Definition 6.1 Let $\mathcal{P} = \{P_n\}_{n\in\mathbb{N}}$ be a sequence of partitions and let $f : \mathbb{N} \to \mathbb{R}$. If $x \in [0,1] \setminus E$, $n \in \mathbb{N}$, and $\epsilon > 0$, we say that $I_n(x) \in P_n$ is *ϵ-good for $\mathcal{P}$ and f* if

$$(1-\epsilon)f(n) < -\log\lambda(I_n(x)) < (1+\epsilon)f(n)$$

or, equivalently,

$$e^{-(1+\epsilon)f(n)} < \lambda(I_n(x)) < e^{-(1-\epsilon)f(n)}.$$

Notation 6.2 In the proofs given in this section, for each $i = 1, 2$, we say that $I_n^i(x)$ is *ϵ-good* if $I_n^i(x)$ is ϵ-good for $\mathcal{P}^i$ and f_i.

Definition 6.3 If $f : \mathbb{N} \to \mathbb{R}$ is a nondecreasing function such that $f(n)$ tends to $+\infty$ as $n \to \infty$, we denote by $f^{[-1]}$ the function $f^{[-1]} : \mathbb{R} \to \mathbb{N}$ defined as $f^{[-1]}(y) = \min\{n \in \mathbb{N} : f(n) \geq y\}$.

Remark 6.4 All the following assertions are immediate consequences of the definition:

(i) $f^{[-1]}$ is nondecreasing;
(ii) $f^{[-1]}(y) \to \infty$ as $y \to +\infty$;

(iii) for each $y \in \mathbb{R}$, $f(f^{[-1]}(y)) \geq y$;
(iv) if $n \in \mathbb{N}$ and $y \in \mathbb{R}$ are such that $n < f^{[-1]}(y)$, then $f(n) < y$; and
(v) if f is strictly increasing and $\tilde{f} : [0, +\infty) \to \mathbb{R}$ is any strictly increasing function such that $\tilde{f}(n) = f(n)$ for each $n \in \mathbb{N}$, then $f^{[-1]}(y) = \lceil \tilde{f}^{-1}(y) \rceil$ for every $y \in \mathbb{R}$.

The following lemma tells us that, if f is not allowed to grow too fast, we can reverse $f^{[-1]}$ by f asymptotically.

Lemma 6.5 *Let $f : \mathbb{N} \to \mathbb{R}$ be a nondecreasing function tending to infinity. If $f(n+1) - f(n) = o(f(n))$ as $n \to \infty$, then $f(f^{[-1]}(y))/y \to 1$ as $y \to +\infty$ (over $y \in \mathbb{R}$).*

Proof By Remark 6.4 (ii), $f^{[-1]}(y) \to \infty$ as $y \to +\infty$. Thus, the fact that $f(n+1) - f(n) = o(f(n))$ as $n \to \infty$ implies $f(f^{[-1]}(y))/f(f^{[-1]}(y)-1) \to 1$ as $y \to +\infty$. Moreover, by items (iii) and (iv) of Remark 6.4, $f(f^{[-1]}(y)-1) < y \leq f(f^{[-1]}(y))$. Therefore, dividing through by $f(f^{[-1]}(y))$ implies the result, as both extremes tend to 1 as $y \to +\infty$. □

6.2 Lochs-type theorem a.e.

In this section, we prove our general Lochs-type theorem a.e., namely Theorem 6.8. This result is a consequence of Propositions 6.6 and 6.7.

Proposition 6.6 *Let $\mathcal{P}^1$ and $\mathcal{P}^2$ be two a.e. log-balanced sequences of partitions with weight functions f_1 and f_2, respectively, with respect to some Borel probability measure λ. If f_2 is nondecreasing, then*

$$\limsup_{n\to\infty} \frac{f_2(L_n(x, \mathcal{P}^1, \mathcal{P}^2))}{f_1(n)} \leq 1 \quad a.e.\ (\lambda).$$

Proof Let $\epsilon \in (0, 1)$. For each $n \in \mathbb{N}$, let $m(n) := f_2^{[-1]}\big(\frac{1+\epsilon}{1-\epsilon} f_1(n)\big)$. Let $m \in \mathbb{N}$ such that $m \geq m(n)$. Since f_2 is nondecreasing, $f_2(m) \geq f_2(m(n)) \geq \frac{1+\epsilon}{1-\epsilon} f_1(n)$, by Remark 6.4 (iii). Thus, $e^{-(1+\epsilon)f_1(n)} \geq e^{-(1-\epsilon)f_2(m)}$ for all $n \in \mathbb{N}$. Hence, for each $x \in [0, 1] \setminus (E^1 \cup E^2)$, each $n \in \mathbb{N}$, and each $m \in \mathbb{N}$ such that $m \geq m(n)$, if $I_n^1(x)$ and $I_m^2(x)$ are ϵ-good, then

$$\lambda(I_n^1(x)) > e^{-(1+\epsilon)f_1(n)} \geq e^{-(1-\epsilon)f_2(m)} > \lambda(I_m^2(x))$$

and, in particular, $I_n^1(x) \nsubseteq I_m^2(x)$. Therefore, for each $x \in [0, 1] \setminus (E^1 \cup E^2)$ and each $n \in \mathbb{N}$ such that both $I_n^1(x)$ is ϵ-good and $I_m^2(x)$ is ϵ-good for each $m \geq m(n)$, we have that $L_n(x, \mathcal{P}^1, \mathcal{P}^2) < m(n)$.

Since $-\log \lambda(I_n^1(x))/f_1(n) \to 1$ as $n \to \infty$ a.e., we have that $I_n^1(x)$ is ϵ-good for n large enough a.e. Notice that, since $f_1(n) \to +\infty$ as $n \to \infty$, Remark 6.4 (ii) implies $m(n) \to \infty$ as $n \to \infty$. Thus, the fact that $-\log \lambda(I_m^2(x))/f_2(m) \to 1$ as $m \to \infty$ a.e. implies that for a.e. x, for n large enough (depending on x), $I_m^2(x)$ is

ϵ-good for each $m \in \mathbb{N}$ such that $m \geq m(n)$. Hence, $L_n(x, \mathcal{P}^1, \mathcal{P}^2) < m(n)$ for n large enough a.e. Therefore, by Remark 6.4 (iv), $f_2(L_n(x, \mathcal{P}^1, \mathcal{P}^2)) < \frac{1+\epsilon}{1-\epsilon} f_1(n)$ for n large enough a.e. Taking $\epsilon \to 0$ gives the result. □

Proposition 6.7 *Let $\mathcal{P}^1$ and $\mathcal{P}^2$ be two a.e. log-balanced sequences of partitions with weight functions f_1 and f_2, respectively, with respect to some Borel probability measure λ such that the following assertions hold:*

(i) f_2 is nondecreasing;
(ii) for each $\eta \in (0, 1)$, there exists $\epsilon > 0$ such that

$$\sum_{n=1}^{\infty} e^{-(1-\epsilon) f_1(n)+(1+\epsilon) f_2(m(n))} < +\infty, \quad \textit{where } m(n) := f_2^{[-1]}((1-\eta) f_1(n)).$$

Then,

$$\liminf_{n\to\infty} \frac{f_2(L_n(x, \mathcal{P}^1, \mathcal{P}^2))}{f_1(n)} \geq 1 \quad \textit{a.e. } (\lambda).$$

Proof Let $\eta \in (0, 1)$. For each $n \in \mathbb{N}$, let $\mathcal{J}_n$ be the set of ordered pairs $(I_n^1(x), I_{m(n)}^2(x))$ such that $I_n^1(x) \not\subseteq I_{m(n)}^2(x)$. Let $(I_n^1(x), I_{m(n)}^2(x)) \in \mathcal{J}_n$. Since $I_n^1(x)$ and $I_{m(n)}^2(x)$ share the point x, necessarily $I_n^1(x)$ contains at least one of the endpoints of $I_{m(n)}^2(x)$. Thus, there are no three ordered pairs in $\mathcal{J}_n$ having the same second entry.

Suppose $\epsilon > 0$ (we will choose ϵ later). Let $\mathcal{J}_{n,\epsilon}$ be the set of ordered pairs in $\mathcal{J}_n$ whose entries are both ϵ-good. Notice that, if $(I_n^1(x), I_{m(n)}^2(x)) \in \mathcal{J}_{n,\epsilon}$, then

$$\lambda(I_n^1(x)) < e^{-(1-\epsilon) f_1(n)} \quad \text{and} \quad e^{-(1+\epsilon) f_2(m(n))} < \lambda(I_{m(n)}^2(x))$$

and, consequently,

$$\lambda(I_n^1(x)) < e^{-(1-\epsilon) f_1(n)+(1+\epsilon) f_2(m(n))} \lambda(I_{m(n)}^2(x)).$$

Let $D_{n,\epsilon} := \{x : \big(I_n^1(x), I_{m(n)}^2(x)\big) \in \mathcal{J}_{n,\epsilon}\} = \bigcup_{(I_n^1(x), I_{m(n)}^2(x)) \in \mathcal{J}_{n,\epsilon}} (I_n^1(x) \cap I_{m(n)}^2(x))$. Hence, this gives

$$\begin{aligned}
\lambda(D_{n,\epsilon}) &= \sum_{(I_n^1(x), I_{m(n)}^2(x)) \in \mathcal{J}_{n,\epsilon}} \lambda(I_n^1(x) \cap I_{m(n)}^2(x)) \leq \sum_{(I_n^1(x), I_{m(n)}^2(x)) \in \mathcal{J}_{n,\epsilon}} \lambda(I_n^1(x)) \\
&\leq e^{-(1-\epsilon) f_1(n)+(1+\epsilon) f_2(m(n))} \sum_{(I_n^1(x), I_{m(n)}^2(x)) \in \mathcal{J}_{n,\epsilon}} \lambda(I_{m(n)}^2(x)) \\
&\leq 2e^{-(1-\epsilon) f_1(n)+(1+\epsilon) f_2(m(n))}.
\end{aligned}$$

(Recall that there are no three ordered pairs in $\mathcal{J}_n$ having the same second entry.)

Let ϵ be as in assertion (ii). Hence, by the Borel–Cantelli Lemma we have that

$$\lambda(\{x : x \in D_{n,\epsilon} \text{ i.o.}\}) = 0.$$

Thus, $x \notin D_{n,\epsilon}$ for n large enough a.e. Since $f_1(n) \to +\infty$ as $n \to \infty$, Remark 6.4 (ii) ensures that $m(n) \to \infty$ as $n \to \infty$. Thus, since f_1 and f_2 are a.e. weight functions of $\mathcal{P}_1$ and $\mathcal{P}^2$, $I_n^1(x)$ and $I_{m(n)}^2(x)$ are ϵ-good for n large enough a.e. By the definition of $D_{n,\epsilon}$, necessarily $I_n^1(x) \subseteq I_{m(n)}^2(x)$ for n large enough a.e. In particular, $L_n(x, \mathcal{P}^1, \mathcal{P}^2) \geq m(n)$ for n large enough a.e. By assertion (i), the fact that $f_2(n) \to +\infty$ as $n \to \infty$, and Remark 6.4 (iii), one has

$$f_2(L_n(x, \mathcal{P}^1, \mathcal{P}^2)) \geq f_2(m(n)) \geq (1-\eta) f_1(n) \quad \text{for } n \text{ large enough a.e.}$$

Taking $\eta \to 0$ gives the result. □

Now we are ready to prove our main result a.e.

Theorem 6.8 *Let $\mathcal{P}^1$ and $\mathcal{P}^2$ be two a.e. log-balanced sequences of partitions with weight functions f_1 and f_2, respectively, with respect to some Borel probability measure λ on $[0, 1]$ such that all the following assertions hold:*

(i) $\sum_{n=1}^{\infty} e^{-\delta f_1(n)} < \infty$ *for every* $\delta > 0$;
(ii) f_2 *is nondecreasing;*
(iii) $f_2(n+1) - f_2(n) = o(f_2(n))$ *as* $n \to \infty$.

Then,

$$\lim_{n\to\infty} \frac{f_2(L_n(x, \mathcal{P}^1, \mathcal{P}^2))}{f_1(n)} = 1 \quad \text{a.e. } (\lambda).$$

Proof By Propositions 6.6 and 6.7, it suffices to check that, for each $\eta \in (0, 1)$, there exists $\epsilon > 0$ so that

$$\sum_{n=1}^{\infty} e^{-(1-\epsilon) f_1(n) + (1+\epsilon) f_2(m(n))} < +\infty, \quad \text{where } m(n) := f_2^{[-1]}\left((1-\eta) f_1(n)\right). \tag{31}$$

Let $\eta > 0$. We show that $\epsilon = \eta/2$ satisfies the condition. By assertion (iii), we can apply Lemma 6.5 to $f = f_2$ and $y = (1-\eta) f_1(n)$ in order to obtain $f_2(m(n))/((1-\eta) f_1(n)) \to 1$ as $n \to \infty$.

Hence, since $\epsilon = \eta/2$, for n large enough one has

$$-(1-\epsilon) f_1(n) + (1+\epsilon) f_2(m(n)) = \left(-\frac{\eta^2}{2} + o(1)\right) f_1(n) \leq -\frac{\eta^2}{4} f_1(n).$$

It follows from assertion (i) that (31) holds. This proves the theorem. □

Remark 6.9 The assumptions of the above result are sufficient conditions. These assumptions are however not always necessary. The next result shows that, for instance, the conclusion of Theorem 6.8 still holds in the case where $\mathcal{P}^1 = \mathcal{P}^2 = \mathcal{F}$ even if the weight function $f_{\mathcal{F}}(n) = 2\log n$ does not satisfy assumption (i) of the theorem.

Proposition 6.10 *Let $\mathcal{P}$ be an a.e. log-balanced sequence of partitions with weight function f with respect to some Borel probability measure λ. If f is nondecreasing, then*

$$\frac{f(L_n(x,\mathcal{P},\mathcal{P}))}{f(n)} = 1 \quad \textit{as } n \to \infty, \quad \textit{a.e. } (\lambda).$$

Proof Notice that, by definition of Lochs' index, $L_n(x,\mathcal{P},\mathcal{P}) \geq n$ for each $n \in \mathbb{N}$. Moreover, since f is nondecreasing, $f(L_n(x,\mathcal{P},\mathcal{P})) \geq f(n)$ for every $n \in \mathbb{N}$. Thus, $\liminf_{n\to\infty} f(L_n(x,\mathcal{P},\mathcal{P}))/f(n) \geq 1$. Therefore, by Proposition 6.6, the proof is complete. □

Remark 6.11 It is unclear whether assumption (ii) in Proposition 6.7 is necessary. Consider in particular the case where $\mathcal{P}^2$ is a slight perturbation of $\mathcal{P}^1$ so that $f_1 = f_2$. A natural question is whether it is possible, in this case, to have $\liminf_{n\to\infty} \frac{f_2(L_n(x,\mathcal{P}^1,\mathcal{P}^2))}{f_1(n)} < 1$ a.e. or that this limit does not even exist. If possible, this situation could be regarded as revealing an intrinsic difficulty, in the sense that considering only the lengths of the intervals does not provide enough information for proving Lochs' conversion results.

6.3 Lochs-type theorem in measure

In this section, we prove our general Lochs-type theorem in measure, namely Theorem 6.14. This result is a consequence of Propositions 6.12 and 6.13.

Proposition 6.12 *Let $\mathcal{P}^1$ and $\mathcal{P}^2$ be two sequences of partitions that are log-balanced in measure with weight functions f_1 and f_2, respectively, with respect to λ, such that the following assertions hold:*

(i) $\mathcal{P}^2$ is self-refining;
(ii) f_2 is nondecreasing.

Then,

$$\limsup_{n\to\infty} \frac{f_2(L_n(x,\mathcal{P}^1,\mathcal{P}^2))}{f_1(n)} \leq 1 \quad \textit{in measure } (\lambda).$$

Proof Let $\epsilon \in (0,1)$ and let $m(n) := f_2^{[-1]}\big(\frac{1+\epsilon}{1-\epsilon} f_1(n)\big)$. Arguing as in the proof of Proposition 6.6, $e^{-(1+\epsilon)f_1(n)} \geq e^{-(1-\epsilon)f_2(m(n))}$ for all $n \in \mathbb{N}$. Hence, for each $x \in [0,1] \setminus (E^1 \cup E^2)$ and each $n \in \mathbb{N}$, if $I_n^1(x)$ and $I_{m(n)}^2(x)$ are ϵ-good, then

$$\lambda(I_n^1(x)) > e^{-(1+\epsilon)f_1(n)} \geq e^{-(1-\epsilon)f_2(m(n))} > \lambda(I_{m(n)}^2(x))$$

and, in particular, $I_n^1(x) \not\subseteq I^2_{m(n)}(x)$. Moreover, as we are assuming that $\mathcal{P}^2$ is self-refining, it follows that $I_n^1(x) \not\subseteq I_m^2(x)$ for each $m \geq m(n)$. Therefore, $L_n(x, \mathcal{P}^1, \mathcal{P}^2) < m(n)$. Hence,

$$\begin{aligned} C_n &:= \{x : L_n(x, \mathcal{P}^1, \mathcal{P}^2) \geq m(n)\} \\ &\subseteq \{x : I_n^1(x) \text{ is not } \epsilon\text{-good}\} \cup \{x : I^2_{m(n)}(x) \text{ is not } \epsilon\text{-good}\}. \end{aligned}$$

By Remark 6.4 (ii), $m(n) \to \infty$ as $n \to \infty$. Thus, the fact that f_1 and f_2 are weight functions in measure implies that, for n large enough,

$$\lambda(\{x : I_n^1(x) \text{ is } \epsilon\text{-good}\}) > 1\text{-}\epsilon \quad \text{and} \quad \lambda(\{x : I^2_{m(n)}(x) \text{ is } \epsilon\text{-good}\}) > 1\text{-}\epsilon.$$

Hence, $\lambda(C_n) \leq 2\epsilon$ for n large enough. Moreover, by Remark 6.4 (iv), if $x \in [0,1]\backslash C_n$, then

$$f_2(L_n(x, \mathcal{P}^1, \mathcal{P}^2)) < \frac{1+\epsilon}{1-\epsilon} f_1(n) = \left(1 + \frac{2\epsilon}{1-\epsilon}\right) f_1(n).$$

Taking $\epsilon \to 0$ gives the result. □

Proposition 6.13 *Let $\mathcal{P}^1$ and $\mathcal{P}^2$ be two sequences of partitions that are log-balanced in measure with weight functions f_1 and f_2, respectively, with respect to a measure λ, such that the following assertions hold:*

(i) f_2 is nondecreasing;
(ii) for each $\eta \in (0, 1)$, there exists some $\epsilon > 0$ such that

$$\lim_{n\to\infty} (1-\epsilon) f_1(n) - (1+\epsilon) f_2(m(n)) = +\infty, \textit{ where } m(n) := f_2^{[-1]}((1-\eta) f_1(n)).$$

Then,

$$\liminf_{n\to\infty} \frac{f_2(L_n(x, \mathcal{P}^1, \mathcal{P}^2))}{f_1(n)} \geq 1 \quad \textit{in measure } (\lambda).$$

Proof Let $\eta \in (0, 1)$ and let ϵ be as in (ii). Let $D_{n,\epsilon}$ be defined as in the proof of Theorem 6.7. As argued in that proof, $\lambda(D_{n,\epsilon}) \leq 2e^{-(1-\epsilon) f_1(n) + (1+\epsilon) f_2(m(n))}$. Because of the choice of ϵ, $\lambda(D_{n,\epsilon}) < \eta$ for n large enough. Notice that, since $f_1(n) \to +\infty$ as $n \to \infty$, Remark 6.4 (ii) ensures that $m(n) \to \infty$ as $n \to \infty$. As f_1 and f_2 are weight functions in measure for $\mathcal{P}^1$ and $\mathcal{P}^2$, then, for n large enough,

$$\lambda(\{x : I_n^1(x) \text{ is } \epsilon - \text{good}\}) > 1 - \eta \quad \text{and} \quad \lambda(\{x : I^2_{m(n)}(x) \text{ is } \epsilon - \text{good}\}) > 1 - \eta.$$

Hence, since

$$\begin{aligned} B_n &:= \{x : L_n(x, \mathcal{P}^1, \mathcal{P}^2) < m(n)\} \\ &\subseteq \{x : I_n^1(x) \not\subseteq I^2_{m(n)}(x)\} \end{aligned}$$

$$\subseteq D_{n,\epsilon} \cup \{x : I_n^1(x) \text{ is not } \epsilon\text{-good}\} \cup \{x : I_{m(n)}^2(x) \text{ is not } \epsilon\text{-good}\},$$

we have that $\lambda(B_n) < 3\eta$ for n large enough. Moreover, by assertion (i), the fact that $f_2(n) \to +\infty$ as $n \to \infty$, and Remark 6.4 (iii), if $x \in [0, 1] \setminus B_n$, then

$$f_2(L_n(x, \mathcal{P}^1, \mathcal{P}^2)) \geq f_2(m(n)) \geq (1 - \eta) f_1(n)$$

for n large enough. Taking $\eta \to 0$ gives the result. □

Theorem 6.14 *Let $\mathcal{P}^1$ and $\mathcal{P}^2$ be two sequences of partitions that are log-balanced in measure, having respective weight functions f_1 and f_2, with respect to some Borel probability measure λ on $[0, 1]$ such that all the following assertions hold:*

(i) $\mathcal{P}^2$ is self-refining;
(ii) f_2 is nondecreasing;
(iii) $f_2(n + 1) - f_2(n) = o(f_2(n))$ as $n \to \infty$.

Then,

$$\lim_{n\to\infty} \frac{f_2(L_n(x; \mathcal{P}^1, \mathcal{P}^2))}{f_1(n)} = 1 \quad \textit{in measure } (\lambda).$$

Proof Arguing as in the proof of Theorem 6.8, for each $\eta > 0$, if $\epsilon = \eta/2$, then $(1 - \epsilon) f_1(n) - (1 + \epsilon) f_2(m(n)) \geq \frac{\eta^2}{4} f_1(n)$ for n large enough. As $f_1(n) \to +\infty$ as $n \to \infty$, the theorem follows by Propositions 6.12 and 6.13. □

Remark 6.15 A situation analogous to that pointed out in Remark 6.9 holds also in connection with the above theorem, that is, the assumptions in Theorem 6.14 are sufficient but not always necessary. Indeed, reasoning as in the proof of Proposition 6.10, one can prove that if $\mathcal{P}$ is log-balanced in measure and self-refining, then $L_n(x, \mathcal{P}, \mathcal{P})/n \to 1$ in measure as $n \to \infty$.

7 Final remarks and open questions

Theorem 6.8 and 6.14 provide asymptotic results for Lochs' indexes for log-balanced sequences of partitions. While the assumption of log-balancedness could be regarded as relatively mild, it is crucial for our results. A natural question is whether we can express sufficient conditions for log-balancedness, for instance in dynamical terms in the case of fibred systems.

We have proved Lochs-type statements a.e. and in measure. Further studies concerning these Lochs-type results could be conducted, such as a multifractal analysis in the spirit of [5], or probabilistic estimates (central limit theorems) such as in [20, 46].

One motivation for the present work comes from the experimental simulation of sources determined by sequences of partitions. (Here, by a source we mean the usual notion in information theory.) Lochs-type results come up naturally when we wonder how many of the initial digits in the binary expansion (which is computationally natural) of a random number are needed in order to deduce a prefix of a given length expressed in a different numeration system. Let us first illustrate this with the following question: how many binary digits of a random number do we need in order to generate the prefix of length n of the characteristic Sturmian word (see Sect. 4.6.2 for the definition)? We gave an answer to this question in Sect. 2.2.2.

Our work is also related to the probabilistic study of the behavior of data structures which are of fundamental importance in computer science. One such structure is the so-called *trie*. A trie is a tree-like data structure that mimics the natural strategy to search a word in a dictionary by comparing words via their prefixes. Thus, the depth of a trie built from a set of words is related to "coincidences" between the words, that is, to the difficulty in distinguishing these words. For positive entropy, Clément, Flajolet and Vallée [13] show that, when one builds a trie by picking N random words produced by a reasonably well-made source with entropy $h > 0$, its average depth has expectation asymptotically equal to $(1/h) \log N$. Furthermore, Cesaratto and Vallée [11] have shown that its distribution is asymptotically Gaussian. In [8] the authors consider the average depth of the tries produced by random words produced by a source also as a mean to compare zero entropy sources.

We conjecture that there should be some relation between the weight function f, when it exists, and the average depth of the trie. We observe that the average depth of a trie is related to the inverse of the weight function, for instance, in the case of positive entropy and for the Farey sequence. Note that $f^{-1}(\log N) = (1/h) \log N$, in the case of positive entropy h, as then we have the weight function $f(n) = hn$. For the case of the Farey sequence of partitions, $f(n) = 2 \log n$ and so $f^{-1}(\log N) = \sqrt{N}$. This coincides with the asymptotics for the expected value of the average depth of the Farey trie given in [8] up to a constant factor. However, this kind of connection does not hold for Stern–Brocot. Very recently, some steps in this line of research have been taken in [2].

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