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Sphere and projective space of a $C^{\star}$-algebra with a faithful state

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Abstract: Let $\mathcal{A}$ be a unital $C^{\star}$-algebra with a faithful state φ. We study the geometry of the unit sphere $\mathbb{S}_{\varphi} = \{x \in \mathcal{A} : \varphi(x^{\star}x) = 1\}$ and the projective space $\mathbb{P}_{\varphi} = \mathbb{S}_{\varphi}/\mathbb{T}$. These spaces are shown to be smooth manifolds and homogeneous spaces of the group $\mathcal{U}_{\varphi}(\mathcal{A})$ of isomorphisms acting in $\mathcal{A}$ which preserve the inner product induced by φ, which is a smooth Banach-Lie group. An important role is played by the theory of operators in Banach spaces with two norms, as developed by M.G. Krein and P. Lax. We define a metric in $\mathbb{P}_{\varphi}$, and prove the existence of minimal geodesics, both with given initial data, and given endpoints.

Keywords: homogeneous space, minimal curves, $C^{\star}$-algebra, projective space

MSC: 46L30, 58B20

1 Introduction

Let $\mathcal{A}$ be a unital $C^{\star}$-algebra with a faithful state φ. There are natural geometric objects associated to this pair: the *unit sphere*

$$\mathbb{S}_{\varphi} := \{x \in \mathcal{A} : \varphi(x^{\star}x) = 1\},$$

and the *projective space*

$$\mathbb{P}_{\varphi} := \mathbb{S}_{\varphi}/\mathbb{T},$$

where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, i.e., $x, x' \in \mathbb{S}_{\varphi}$ define the same element in $\mathbb{P}_{\varphi}$ if $x' = zx$ for some $z \in \mathbb{T}$.

The purpose of this paper is the study of these spaces using the tools of differential geometry. As in the classical setting in finite dimension (the spaces $\mathbb{S}_{\varphi}$ and $\mathbb{P}_{\varphi}$ are infinite dimensional), a key feature in this study is the transitive action of a group of movements. In this case, the group $\mathcal{U}_{\varphi}(\mathcal{A})$ of invertible linear operators acting in $\mathcal{A}$, which preserve the inner product $\langle\ ,\ \rangle_{\varphi}$ induced by φ ($\langle x, y\rangle_{\varphi} = \varphi(y^{\star}x)$, $x, y \in \mathcal{A}$) , i.e.,

$$\mathcal{U}_{\varphi}(\mathcal{A}) = \{G \in \mathcal{B}(\mathcal{A}) : G \text{ is invertible and } \langle Gx, Gy\rangle_{\varphi} = \langle x, y\rangle_{\varphi}\}.$$

Here $\mathcal{B}(\mathcal{A})$ denotes the Banach space of bounded linear operators acting in $\mathcal{A}$.

The sphere $\mathbb{S}_{\varphi}$ is a dense subset of a sphere in a Hilbert space: denote by $\mathcal{L} = L^2(\mathcal{A}, \varphi)$ the GNS Hilbert space of the pair $(\mathcal{A}, \varphi)$, i.e. the completion of the pre-Hilbert space $(\mathcal{A}, \langle\ ,\ \rangle_{\varphi})$. Then clearly $\mathbb{S}_{\varphi}$ is dense in the sphere of $\mathcal{L}$. Also it is clear that an element $G \in \mathcal{U}_{\varphi}(\mathcal{A})$ extends uniquely to a unitary operator U_G in $\mathcal{L}$. Therefore $\mathcal{U}_{\varphi}(\mathcal{A})$ can be regarded as a subgroup of the unitary group of $\mathcal{L}$, consisting of all unitaries U acting in $\mathcal{L}$ which leave $\mathcal{A}$ fixed: $U(\mathcal{A}) = \mathcal{A}$. To perform our geometric study, we shall need to introduce topologies in $\mathbb{S}_{\varphi}$ and $\mathbb{P}_{\varphi}$ (the ambient topology of $\mathcal{A}$, and its quotient topology, respectively), and also in $\mathcal{U}_{\varphi}(\mathcal{A})$. Clearly, $\mathcal{U}_{\varphi}(\mathcal{A})$ is not a Banach-Lie group in the topology that it inherits from the whole unitary group of $\mathcal{L}$: the condition of leaving $\mathcal{A} \subset \mathcal{L}$ fixed is not closed ($\mathcal{A}$ is dense in $\mathcal{L}$). To obtain a regular structure for $\mathcal{U}_{\varphi}(\mathcal{A})$ we shall use the theory of operators in spaces with two norms, developed independently by M.G. Krein [1] and P. Lax [2]. Two

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norms appear naturally in our context, the usual norm $\|\cdot\|$ of $\mathcal{A}$ and the norm $\|\cdot\|_\varphi$ induced by the φ-inner product.

The projective space $\mathbb{P}_\varphi$ is homeomorphic to a set of projections, the space $\mathcal{P}_1(\mathcal{A},\varphi)$ of *rank one* projections acting in $\mathcal{A}$, which are orthogonal for the φ-inner product. We introduce a natural metric in $\mathbb{P}_\varphi$, and study its geodesics. Though the metric is not complete (and the space is infinite dimensional), using facts from the infinite dimensional Grassmann manifold [3, 4], we obtain the existence of minimal geodesics with

1. fixed initial position and initial velocity (**Theorem 6.2**);
2. fixed endpoints (**Theorem 6.3**).

The contents of the paper are the following. In Sections 2 and 3 we introduce preliminary facts; Section 3 focuses on basic facts concerning operators in spaces with two norms (our references here are [1] and [2], and also the paper by I. C. Gohberg and M. K. Zambickii [5]), and the local structure of the group $\mathcal{U}_\varphi(\mathcal{A})$. In Section 3 we study the actions of $\mathcal{U}_\varphi(\mathcal{A})$ on $\mathbb{S}_\varphi$ and $\mathbb{P}_\varphi$. Our main result is that both actions are transitive, and that the sphere and projective space are connected. In Section 4 we examine the local regular structure of $\mathbb{S}_\varphi$ and $\mathbb{P}_\varphi$: the first space is a complemented submanifold of $\mathcal{A}$, the second is a differentiable manifold, both spaces are homogeneous spaces of the group $\mathcal{U}_\varphi(\mathcal{A})$. In Section 5 we introduce a pre-Riemann-Hilbert metric in $\mathbb{P}_\varphi$. We do this in two ways, that turn out to coincide: as a quotient metric, and as a trace induced metric. In Section 6 we prove the existence of minimal geodesics for this metric, both in the initial value problem (given initial position and velocity) and in the boundary value problem (given initial and final position).

2 Preliminaries and notation

We shall consider $\mathcal{A}$ represented in the Hilbert space $\mathcal{L}=L^2(\mathcal{A},\varphi)$, via the GNS representation induced by φ. Elements $x\in\mathcal{A}$ will also be regarded as elements of $\mathcal{L}$ with norm $\|x\|_\varphi=\varphi(x^*x)^{1/2}$. As usual, if $x,y\in\mathcal{A}$, $x\otimes y$ will denote the rank one operator acting in $\mathcal{L}$: $x\otimes y(\xi)=\langle\xi,y\rangle x$, and in particular if $a\in\mathcal{A}$, $x\otimes y(a)=\varphi(y^*a)x$. Denote by $\mathcal{P}(\mathcal{L})$ the space of selfadjoint projections of $\mathcal{L}$, and by $\mathcal{P}_1(\mathcal{L})$ the subset of rank one projections. Let the map

$$\pi:\mathbb{S}_\varphi\to\mathcal{P}_1(\mathcal{L})\,,\quad \pi(x)=x\otimes x$$

whose range is $\mathcal{P}_1(\mathcal{A},\varphi)$ the set of rank one projections onto lines generated by elements in $\mathcal{A}\subset\mathcal{L}$. This map induces a bijection

$$\mathbb{P}_\varphi\to\mathcal{P}_1(\mathcal{A},\varphi)\,,\quad [x]\to x\otimes x.$$

We shall identify these spaces (we shall see that the bijection above is a homeomorphism between the quotient topology and the norm topology of bounded operators acting in $\mathcal{A}$). Therefore the map π can also be regarded as the quotient map from $\mathbb{S}_\varphi$ onto $\mathbb{P}_\varphi$.

Part of the material in this section is either well known or follows from well known facts. It falls in the context of the theory of symmetrizable and proper linear operators in spaces with two norms, developed by I. Gohberg and M.K. Zambickii [5], M.G. Krein [1] and P. Lax [2]. In the space $\mathcal{A}$ we can consider two norms, the usual norm $\|\cdot\|_\infty$ and the norm $\|\cdot\|_\varphi$ induced by φ. These norms are comparable: $\|a\|_\varphi\le\|a\|_\infty$ for all $a\in\mathcal{A}$, and only the second norm is complete.

A bounded linear operator $T\in\mathcal{B}(\mathcal{A})$ is *adjointable* if there exists $S\in\mathcal{B}(\mathcal{A})$ such that

$$\varphi(y^*T(x))=\varphi(S(y)^*x).$$

In this case we denote $S=T^\sharp$. For example, if L_a denotes the left multiplication operator ($a\in\mathcal{A}$) then $L_a^\sharp=L_{a^*}$. Let us denote by

$$\mathcal{B}_{\mathbf{a}}(\mathcal{A})=\{T\in\mathcal{B}(\mathcal{A}):T\text{ is adjointable}\}.$$

Note that $\mathcal{B}_{\mathbf{a}}(\mathcal{A})$ is a (non closed) subalgebra of $\mathcal{B}(\mathcal{A})$. We shall consider a subset of $\mathcal{B}_{\mathbf{a}}(\mathcal{A})$

$$\mathcal{B}_s(\mathcal{A})=\{T\in\mathcal{B}_{\mathbf{a}}(\mathcal{A}):T^\sharp=T\}.$$

the adjointable operators which are symmetric with respect to φ. These operators are called usually *symmetrizable*. M.G. Krein [1] and P. Lax [2] studied this class, in the context of a Banach algebra $\mathcal{B}$ (here equal to $\mathcal{A}$) endowed with a positive definite inner product (here, the one induced by the state φ). For instance, they showed that these operators extend to selfadjoint operators in $\mathcal{L}$ (we state below a result independently obtained by both authors). Adjointable operators also extend to $\mathcal{L}$. If T is adjointable, $T_1 = \frac{1}{2}(T + T^\sharp)$ and $T_2 = \frac{-i}{2}(T - T^\sharp)$ are symmetrizable, and therefore extend to $\mathcal{L}$, thus $T = T_1 + iT_2$ extends to $\mathcal{L}$, as well as $T^\sharp$, and clearly the extension of $T^\sharp$ is the adjoint in $\mathcal{L}$ of the extension of T. This latter result was obtained by I. Gohberg and M.K. Zambickii [5] in the much broader context of Banach spaces with two norms (none of them given by inner products).

Let $\mathcal{F}(\mathcal{A})$ be the linear span of $a \otimes b$, $a, b \in \mathcal{A}$ in $\mathcal{B}(\mathcal{A})$, which can be regarded also as finite rank operators acting in $\mathcal{L}$, with symbols in $\mathcal{A}$. These operators are adjointable, $(x \otimes y)^\sharp = y \otimes x$. Put

$$\mathcal{F}(\mathcal{A})_s = \{T \in \mathcal{F}(\mathcal{A}) : \varphi(x^* Ty) = \varphi((Tx)^* y)\},$$

the subset of operators in $\mathcal{F}(\mathcal{A})$ which are symmetric for the φ-inner product. Namely, $\mathcal{F}(\mathcal{A})_s = \mathcal{F}(\mathcal{A}) \cap \mathcal{B}_s(\mathcal{A})$.

We introduce a norm in $\mathcal{B}_{\mathbf{a}}(\mathcal{A})$:

$$\|T\|_{\mathbf{a}} = \max\{\|T\|, \|T^\sharp\|\}.$$

This norm was introduced by Gohberg and Zambickii in [5]. It is easy to verify that $\mathcal{B}_{\mathbf{a}}(\mathcal{A})$ is complete with this norm. Also that

$$\|TS\|_{\mathbf{a}} \le \|T\|_{\mathbf{a}} \|S\|_{\mathbf{a}},$$

i.e. $\mathcal{B}_{\mathbf{a}}(\mathcal{A})$ is a Banach algebra, with involution $\sharp$. Also it is clear that $\|T^\sharp\|_{\mathbf{a}} = \|T\|_{\mathbf{a}}$. Though $\mathcal{B}_{\mathbf{a}}(\mathcal{A})$ is not a C^*-algebra. For instance, pick $a \in \mathcal{A}$ with $\|a\| = 1$ and $a^* a \neq 1$. Elementary computations show that

$$\|(1 \otimes a)^\sharp (1 \otimes a)\|_{\mathbf{a}} = \varphi(a^* a) \text{ and } \|1 \otimes a\|_{\mathbf{a}} \ge \|a\| = 1,$$

where $\varphi(a^* a) < 1$. Indeed, since $a^* a \le 1$ and φ is faithful, $\varphi(a^* a) = 1$ would imply $a^* a = 1$.

Let us denote by $Gl_{\mathbf{a}}(\mathcal{A})$ the group of invertible operators in $\mathcal{B}_{\mathbf{a}}(\mathcal{A})$. Note that $G \in Gl_{\mathbf{a}}(\mathcal{A})$ if only if G and $G^\sharp$ are invertible in $\mathcal{A}$. Further, we have $\sigma_{\mathcal{B}_{\mathbf{a}}(\mathcal{A})}(T) = \sigma_{\mathcal{A}}(T^\sharp) \cup \overline{\sigma_{\mathcal{A}}(T)}$ for $\mathcal{B}_{\mathbf{a}}(\mathcal{A})$.

Consider the closed subgroup

$$\mathcal{U}_\varphi(\mathcal{A}) = \{G \in Gl_{\mathbf{a}}(\mathcal{A}) : \varphi((Gx)^* Gy) = \varphi(x^* y)\}.$$

That is, $\mathcal{U}_\varphi(\mathcal{A})$ consists of the invertible operators acting in $\mathcal{A}$ which preserve the inner product given by the state φ. Namely, if $G \in \mathcal{U}_\varphi(\mathcal{A})$ then $G^{-1} = G^\sharp$.

Elements G in $\mathcal{U}_\varphi(\mathcal{A})$ need not be isometric, and it is clear that $\|G\|_{\mathbf{a}} \ge 1$. Clearly, for $a \in \mathcal{A}$, $\|L_a\|_{\mathbf{a}} = \|a\|_\infty$.

Let $\mathcal{Q}_{\mathbf{a}}$ the set of idempotents in $\mathcal{B}_{\mathbf{a}}(\mathcal{A})$,

$$\mathcal{Q}_{\mathbf{a}} = \{Q \in \mathcal{B}_{\mathbf{a}}(\mathcal{A}) : Q^2 = Q\}.$$

In particular, $\mathcal{Q}_{\mathbf{a}}$ is an analytic submanifold of $\mathcal{B}_{\mathbf{a}}(\mathcal{A})$ (see [6]). We shall consider a subset of $\mathcal{Q}_{\mathbf{a}}$:

$$\mathcal{P}_{\mathbf{a}} = \{P \in \mathcal{Q}_{\mathbf{a}} : P^\sharp = P\},$$

the idempotents which are orthogonal with respect to φ (and extend to selfadjoint projections in $\mathcal{L}$). Recall that $\mathcal{P}_1(\mathcal{A}, \varphi)$ denotes the subset of rank one projections onto lines generated by elements in $\mathcal{A} \subset \mathcal{L}$.

Note that $\mathcal{U}_\varphi(\mathcal{A})$ acts in $\mathcal{P}_{\mathbf{a}}$:

$$G \cdot P = GPG^{-1} \in \mathcal{P}_{\mathbf{a}}, \text{ for } G \in \mathcal{U}_\varphi(\mathcal{A}) \text{ and } P \in \mathcal{P}_{\mathbf{a}}.$$

Before we finish this section, let us state the following elementary result. Note that $\mathcal{P}_1(\mathcal{A}, \varphi)$ is considered with the $\| \, \|_{\mathbf{a}}$-topology.

Lemma 2.1. *The bijection*

$$\mathbb{P}_\varphi \longleftrightarrow \mathcal{P}_1(\mathcal{A}, \varphi), \quad [x] \to x \otimes x, \quad (x \in \mathbb{S}_\varphi)$$

is a homeomorphism.

Proof. First note that an element $x \in \mathcal{A}$ defines a projection $x \otimes x$ in $\mathcal{A}$ if and only if $\varphi(x^*x) = 1$. Therefore, it is clear that the map above is a bijection. It is continuous: if $[x_n] \to [x]$ in $\mathbb{P}_\varphi$, then there exist $z_n \in \mathbb{T}$ such that $z_n x_n \to x$ in $\mathcal{A}$. Then clearly $x_n \otimes x_n = z_n x_n \otimes z_n x_n \to x \otimes x$ in $\mathcal{B}_{\mathbf{a}}(\mathcal{A})$. Suppose now that $x_n, x \in \mathbb{P}_\varphi$ satisfy $x_n \otimes x_n \to x \otimes x$ in $\mathcal{B}_{\mathbf{a}}(\mathcal{A})$. Then

$$x_n \otimes x_n(x) = \varphi(x_n^* x)x_n \to x \otimes x(x) = x,$$

and

$$\langle x_n \otimes x_n(x), x\rangle_\varphi = |\varphi(x_n^* x)|^2 \to \langle x \otimes x(x), x\rangle_\varphi = |\varphi(x^* x)|^2 = 1,$$

i.e. $|\varphi(x_n^* x)| \to 1$. Then putting $z_n = \frac{\varphi(x_n^* x)}{|\varphi(x_n^* x)|} \in \mathbb{T}$, one has that $z_n x_n \to x$ in $\mathcal{A}$. □

In order to study the structure of $\mathcal{P}_{\mathbf{a}}$, we shall need the following elementary facts, which are consequences of the holomorphic functional calculus in the Banach algebra $\mathcal{B}_{\mathbf{a}}(\mathcal{A})$. These facts hold in the broader frame of Banach algebras with involution.

First we consider a local polar decomposition in $Gl_{\mathbf{a}}(\mathcal{A})$. In what follow, we denote by

$$\log(A) := \sum_{n\geq 1} \frac{1}{n}(1 - A)^n,$$

defined for $A \in \mathcal{B}_{\mathbf{a}}(\mathcal{A})$ such that $\|A - 1\|_{\mathbf{a}} < 1$.

Lemma 2.2. *Let $G \in Gl_{\mathbf{a}}(\mathcal{A})$ be close to 1 so that $\|G^\sharp G - 1\|_{\mathbf{a}} < 1$. Then there exist $H \in Gl_{\mathbf{a}}(\mathcal{A})$, $H^\sharp = H$, $H^2 = G^\sharp G$, and $U \in \mathcal{U}_\varphi(\mathcal{A})$, which are C^∞ functions in terms of G, such that*

$$G = UH.$$

Proof. Since $\|G^\sharp G - 1\|_{\mathbf{a}} < 1$, the log series

$$\sum_{n\geq 1} \frac{1}{n}(1 - G^\sharp G)^n = L$$

converges in $\mathcal{B}_{\mathbf{a}}(\mathcal{A})$. Put $H = e^{\frac{1}{2}L}$. Then it is clear that $H^\sharp = H \in Gl_{\mathbf{a}}(\mathcal{A})$ and $H^2 = G^\sharp G$. Also H is a C^∞ function of G, and commutes with $G^\sharp G$.

Put $U = GH^{-1}$. Since $G, H \in Gl_{\mathbf{a}}(\mathcal{A})$, then $U \in Gl_{\mathbf{a}}(\mathcal{A})$ and

$$U^\sharp U = (GH^{-1})^\sharp GH^{-1} = H^{-1}G^\sharp GH^{-1} = 1,$$

i.e. $U \in \mathcal{U}_\varphi(\mathcal{A})$. □

Lemma 2.3. *There exists $0 < r < 1$ such that if $U \in \mathcal{U}_\varphi(\mathcal{A})$ with $\|U - 1\|_{\mathbf{a}} < r$, then there exists $Z \in \mathcal{B}_{\mathbf{a}}(\mathcal{A})$ with $Z^\sharp = -Z$, Z a C^∞ function of U, such that*

$$U = e^Z.$$

Proof. Put

$$Z = \log(U) = \sum_{n\geq 1} \frac{1}{n}(1 - U)^n,$$

which converges in $\mathcal{B}_{\mathbf{a}}(\mathcal{A})$. Clearly $e^Z = U$. Also, since $\|U^\sharp - 1\|_{\mathbf{a}} = \|U - 1\|_{\mathbf{a}} < 1$,

$$Z^\sharp = \sum_{n\geq 1} \frac{1}{n}(1 - U^\sharp)^n = \sum_{n\geq 1} \frac{1}{n}(1 - U^{-1})^n = \log(U^{-1}).$$

Then $e^{Z^\sharp} = U^{-1} = e^{-Z}$, i.e., since Z and $Z^\sharp$ commute, $e^{Z^\sharp + Z} = 1$. It follows that

$$Z^\sharp + Z = \sum_{k=1}^{m} 2k\pi i Q_k, \tag{1}$$

over a finite set of integers k, for $Q_k \in \mathcal{Q}_{\mathbf{a}}$, with $Q_k Q_{k'} = 0$ if $k \neq k'$. Note that

$$\|Z\|_{\mathbf{a}} \leq \sum_{n\geq 1} \frac{1}{n}\|1-U\|_{\mathbf{a}}^n < \sum_{n\geq 1} \frac{1}{n} r^n = -\log(1-r).$$

Pick $0 < r < 1$ such that $-\log(1-r) < \pi$, i.e $0 < r < 1 - e^{-\pi}$. Then

$$\|Z^\sharp + Z\|_{\mathbf{a}} \leq \|Z^\sharp\|_{\mathbf{a}} + \|Z\|_{\mathbf{a}} = 2\|Z\|_{\mathbf{a}} < 2\pi.$$

If there exists an integer $k_0 \neq 0$ in the sum (1) such that $Q_{k_0} \neq 0$, then

$$\|Z^\sharp + Z\|_{\mathbf{a}} \geq 2k_0\pi.$$

Indeed, pick $x_0 \in R(Q_{k_0})$ with $\|x_0\|_\infty = 1$. Then

$$\|\sum_{k\geq 1} 2k\pi i Q_k x_0\|_\infty = \|2k_0\pi i Q_{k_0} x_0\|_\infty = 2k_0\pi.$$

It follows that $Z^\sharp = -Z$. □

In particular, this fact above enables one to obtain a local chart near 1 for $\mathcal{U}_\varphi(\mathcal{A})$, defined on a neighbourhood of the origin in

$$\mathcal{B}_{as}(\mathcal{A}) = \{Z \in \mathcal{B}_{\mathbf{a}}(\mathcal{A}) : Z^\sharp = -Z\},$$

via the exponential map, in a standard fashion, as with the usual unitary group of a C^*-algebra.

Corollary 2.4. *The group $\mathcal{U}_\varphi(\mathcal{A})$ is a Banach-Lie C^∞-group, and a complemented submanifold of $\mathcal{B}_{\mathbf{a}}(\mathcal{A})$. Its Banach-Lie algebra is $\mathcal{B}_{as}(\mathcal{A})$.*

Note that, as any smooth Banach Lie group, $\mathcal{U}\varphi(\mathcal{A})$ turns out to be a real analytic Banach Lie group (see e.g. [7]).

We can use these facts and notations to prove that the symmetric part $\mathcal{P}_{\mathbf{a}}$ of $\mathcal{Q}_{\mathbf{a}}$ is a complemented submanifold of $\mathcal{B}_{\mathbf{a}}(\mathcal{A})$:

Proposition 2.5. *$\mathcal{P}_{\mathbf{a}}$ is a C^∞ submanifold of $\mathcal{B}_{\mathbf{a}}(\mathcal{A})$. The action of $\mathcal{U}_\varphi(\mathcal{A})$ on $\mathcal{P}_{\mathbf{a}}$ is locally transitive and has C^∞ local cross sections. In particular, for any fixed $P_0 \in \mathcal{P}_{\mathbf{a}}$, the map*

$$\pi_{P_0} : \mathcal{U}_\varphi(\mathcal{A}) \to \mathcal{U}_\varphi(\mathcal{A}) \cdot P_0\ , \quad \pi_{P_0}(U) = UP_0U^\sharp$$

is a C^∞ submersion and $\mathcal{U}_\varphi(\mathcal{A}) \cdot P_0$ coincides with the connected component of P_0 in $\mathcal{P}_{\mathbf{a}}$.

We abbreviate these facts in the statement, by saying that the $\mathcal{P}_{\mathbf{a}}$ is a C^∞-homogeneous space of $\mathcal{U}_\varphi(\mathcal{A})$. Along this line, Porta and Recht [6] proved that $\mathcal{Q}_{\mathbf{a}}$ is a homogeneous space of $Gl_{\mathbf{a}}(\mathcal{A})$. To prove this Proposition, we shall need the following result from [8]:

Lemma 2.6. *Let G be a Banach-Lie group acting smoothly on a Banach space X. For a fixed $x_0 \in X$, denote by $\pi_{x_0} : G \to X$ the smooth map $\pi_{x_0}(g) = g \cdot x_0$. Suppose that*

1. *π_{x_0} is an open mapping, regarded as a map from G onto the orbit $\{g \cdot x_0 : g \in G\}$ of x_0 (with the relative topology of X).*
2. *The differential $d(\pi_{x_0})_1 : (TG)_1 \to X$ splits: its nullspace and range are closed complemented subspaces.*

Then the orbit $\{g \cdot x_0 : g \in G\}$ is a smooth submanifold of X, and the map

$$\pi_{x_0} : G \to \{g \cdot x_0 : g \in G\}$$

is a smooth submersion.

Proof. (of Proposition 2.5) In our case, $G = \mathcal{U}_\varphi(\mathcal{A})$, $X = \mathcal{B}_{\mathbf{a}}(\mathcal{A})$, $x_0 = P_0$ and the action $\pi_{P_0} : \mathcal{U}_\varphi(\mathcal{A}) \to \mathcal{B}_{\mathbf{a}}(\mathcal{A})$ is given by conjugation $\pi_{P_0}(U) = UP_0U^{-1}$. Put

$$O_{P_0} = \{UP_0U^\sharp : U \in \mathcal{U}_\varphi(\mathcal{A})\}$$

the orbit of P_0. To prove that the action is locally transitive and that π_{P_0} is an open mapping, we shall construct continuous local cross sections for π_{P_0} defined on a neighbourhood of P_0 in $\mathcal{P}_{\mathbf{a}}$. Consider

$$\mathcal{V}_{P_0} = \{P \in \mathcal{P}_{\mathbf{a}} : G := PP_0 + (1-P)(1-P_0) \in Gl_{\mathbf{a}}(\mathcal{A}) \text{ and } \|G^\sharp G - 1\|_{\mathbf{a}} < 1\}.$$

It is clear that $\mathcal{V}_{P_0}$ is open in $\mathcal{P}_{\mathbf{a}}$: $if P = P_0$ then $G = 1$, thus G is invertible if P is close to P_0. Put

$$s_{P_0} : \mathcal{V}_{P_0} \to \mathcal{U}_\varphi(\mathcal{A}),\ s_{P_0}(P) = U,$$

where $U \in \mathcal{U}_\varphi(\mathcal{A})$ is given by the decomposition $G = UH$ done in Lemma 2.2. Note that s_{P_0} is continuous (the map $G \mapsto U$ is C^∞). Also

$$GP_0 = PP_0 = PG.$$

Thus $G^\sharp P_0 = P_0 G^\sharp$ and $G^\sharp G P_0 = P_0 G^\sharp G$. The operator H is a power series of $G^\sharp G$, thus $HP_0 = P_0 H$. Then

$$UP_0 = GH^{-1}P_0 = GP_0H^{-1} = PGH^{-1} = PU,$$

i.e. s_{P_0} is a cross section for the action. Cross sections on neighbourhoods around other points in $\mathcal{P}_{\mathbf{a}}$ are obtained by translation with the group action.

Let us check condition 2. of Lemma 2.6. These computations are very similar to the case of the Grassmann manifold of $\mathcal{B}(\mathcal{H})$ (see [3]), we include them. We differentiate the map π_{P_0} at $1 \in \mathcal{U}_\varphi(\mathcal{A})$, regarding it as a map to $\mathcal{B}_s(\mathcal{A})$. It is clear that

$$d(\pi_{P_0})_1 : \mathcal{B}_{as}(\mathcal{A}) \to \mathcal{B}_s(\mathcal{A}),\ d(\pi_{P_0})_1(Z) = ZP_0 - P_0Z.$$

Thus the nullspace of $d(\pi_{P_0})_1$ consists of Z in $\mathcal{B}_{as}(\mathcal{A})$ which commute with P_0. Written as 2×2 matrices in terms of P_0, they are the anti-symmetric P_0-diagonal matrices:

$$D_{P_0,as} = \left\{ A \in B_{as}(\mathcal{A}) : A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix},\ \text{where } a = -a^\sharp\ ,\ d = -d^\sharp \right\}.$$

A natural supplement for this nullspace is the space of P_0-co-diagonal anti-symmetric matrices, i.e, $P_0YP_0 = 0 = (1-P_0)Y(1-P_0)$:

$$C_{P_0,as} = \left\{ Y \in B_{as}(\mathcal{A}) : Y = \begin{pmatrix} 0 & -y \\ y^\sharp & 0 \end{pmatrix} \right\}$$

The range of $d(\pi_{P_0})_1$ is $\{ZP_0 - P_0Z : Z \in \mathcal{B}_{as}(\mathcal{A})\}$. This subspace of $\mathcal{B}_s(\mathcal{A})$ coincides with

$$C_{P_0,s} = \left\{ Y \in B_s(\mathcal{A}) : Y = \begin{pmatrix} 0 & y \\ y^\sharp & 0 \end{pmatrix} \right\},$$

the subspace of P_0-co-diagonal symmetric matrices. Indeed, it is clear that the range of $d(\pi_{P_0})_1$ is contained in this subspace. Conversely, pick $Y \in C_{P_0,s}$ and note that

$$YP_0 + P_0Y = \begin{pmatrix} 0 & y \\ y^\sharp & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & y \\ y^\sharp & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ y^\sharp & 0 \end{pmatrix} + \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} = Y$$

Put $Z = YP_0 - P_0Y = \begin{pmatrix} 0 & -y \\ y^\sharp & 0 \end{pmatrix}$. It is clear that $Z^\sharp = -Z$. Note that $d(\pi_{P_0})_1(Z) = Y$. Indeed, put $\gamma(t) = e^{tZ}P_0e^{-tZ}$. Then

$$\begin{aligned} d(\pi_{P_0})_1(Z) = \dot\gamma(0) = \frac{d}{dt}e^{tZ}P_0e^{-tZ}|_{t=0} &= ZP_0 - P_0Z = YP_0 - P_0YP_0 - P_0YP_0 + P_0Y \\ &= YP_0 + P_0Y = Y. \end{aligned}$$

Therefore the range of $d(\pi_{P_0})_1$ is complemented in $\mathcal{B}_s(\mathcal{A})$: a natural supplement is the space of P_0-diagonal symmetric matrices.

Then, by the Lemma 2.6, the orbit O_{P_0} is a smooth submanifold of $B_{\mathbf{a}}(\mathcal{A})$, the map π_{P_0} is a smooth submersion, and $\mathcal{P}_{\mathbf{a}}$ is a discrete union of orbits O_P, $P \in \mathcal{P}_{\mathbf{a}}$. □

3 Group actions in $\mathbb{S}_\varphi$ and $\mathbb{P}_\varphi$

In this section, we shall define natural C^∞ homogeneous structures, induced on $\mathbb{S}_\varphi$ and $\mathbb{P}_\varphi$ by the group action of $\mathcal{U}_\varphi(\mathcal{A})$. Let $x \in \mathbb{S}_\varphi$, and $G \in \mathcal{U}_\varphi(\mathcal{A})$, then the action is given by $G \cdot x = Gx$. Indeed,

$$Gx \in \mathbb{S}_\varphi \text{ and } [Gx] \in \mathbb{P}_\varphi.$$

Operators in $\mathcal{U}_\varphi(\mathcal{A})$ extend to unitary operators in $\mathcal{L}$. Thus one can also regard this subgroup as consisting of unitary operators U acting in $\mathcal{L}$ which satisfy $U(\mathcal{A}) = \mathcal{A}$.

Clearly, $\mathcal{U}_\varphi(\mathcal{A})$ contains $\mathcal{U}_\mathcal{A}$, the unitary group of $\mathcal{A}$, acting by left multiplication on $\mathcal{A}$: $L_u(x) = ux$ ($u \in \mathcal{U}_\mathcal{A}$, $a \in \mathcal{A}$). Also $\mathcal{U}_\varphi(\mathcal{A})$ contains the group of φ-invariant $*$-automorphisms of $\mathcal{A}$: $\theta \in Aut(\mathcal{A})$ such that $\varphi(\theta(x)) = \varphi(x)$, $x \in \mathcal{A}$. For many computations (for instance, to show that the above action is transitive), it will suffice to consider special elements in $\mathcal{U}_\varphi(\mathcal{A})$. For instance, if $X \in \mathcal{F}(\mathcal{A})_s$, we have

$$e^{iX} \in \mathcal{U}_\varphi(\mathcal{A}).$$

Remark 3.1. Let $z \in \mathcal{A}$ with $\varphi(z) = 0$. Put $X = z \otimes 1 + 1 \otimes z$. Then $x \in \mathcal{F}(\mathcal{A})_s$ and

$$e^{iX}(1) = e^{i(z\otimes 1+1\otimes z)}(1) = \cos(\varphi(z^*z)^{1/2}) \cdot 1 + i\frac{\sin(\varphi(z^*z)^{1/2})}{\varphi(z^*z)^{1/2}} \cdot z.$$

Indeed, using that $1 \otimes z(1) = 0 = z \otimes 1(z)$, straightforward computations show that

$$X^{2n}(1) = (z \otimes 1 + 1 \otimes z)^{2n}(1) = \varphi(z^*z)^n \cdot 1$$

and

$$X^{2n+1}(1) = (z \otimes 1 + 1 \otimes z)^{2n+1}(1) = \varphi(z^*z)^n \cdot z.$$

Additionally, in matrix form (in term of $P_0 = 1 \otimes 1$), since

$$X = \begin{pmatrix} 0 & 1 \otimes z \\ z \otimes 1 & 0 \end{pmatrix}$$

then, one has that

$$e^{iX} = \begin{pmatrix} \cos(\varphi(z^*z)^{1/2}) & i(1 \otimes z)\mathrm{sinc}(\varphi(z^*z)^{1/2}) \\ i(z \otimes 1)\mathrm{sinc}(\varphi(z^*z)^{1/2}) & \cos(\varphi(z^*z)^{1/2}) \end{pmatrix} \in \mathcal{U}_\varphi(\mathcal{A}).$$

where $\mathrm{sinc}(t) = \frac{\sin(t)}{t}$ is the cardinal sine, defined for $t \geq 0$.

Lemma 3.2. *Let $y \in \mathbb{S}_\varphi$, $y \notin \mathbb{C}.1$ and $\varphi(y) \neq 0$. Then $|\varphi(y)| < 1$ and*

$$z = -ie^{-i\theta}\frac{\cos^{-1}(|\varphi(y)|)}{(1 - |\varphi(y)|^2)^{1/2}} \cdot (y - \varphi(y))$$

satisfies

$$e^{i(z\otimes 1+1\otimes z)}(1) = e^{-i\theta}y,$$

where $(-\pi, \pi) \ni \theta = \arg(\varphi(y))$.

Proof. First note that since $\varphi(y^*y) = 1$, if $|\varphi(y)| = 1$ then, in the Cauchy-Schwarz inequality

$$1 = |\varphi(y)| \leq \varphi(y^*y)^{1/2}\varphi(1)^{1/2} = 1$$

one would have equality, which would imply $y = \lambda \cdot 1$. Note also that $\varphi(z) = 0$. Therefore, by the above remark,

$$e^{i(z\otimes 1+1\otimes z)}(1) = \cos(\varphi(z^*z)^{1/2}) \cdot 1 + i\frac{\sin(\varphi(z^*z)^{1/2})}{\varphi(z^*z)^{1/2}} \cdot z.$$

Note that

$$z^\star z = \frac{(\cos^{-1}(|\varphi(y)|)^2}{1 - |\varphi(y)|^2}(y^\star - \overline{\varphi(y)})(y - \varphi(y)),$$

and that

$$\varphi((y^\star - \overline{\varphi(y)})(y - \varphi(y))) = 1 - |\varphi(y)|^2.$$

Thus $\varphi(z^\star z) = (\cos^{-1}(|\varphi(y)|))^2$ and therefore

$$e^{i(z\otimes 1+1\otimes z)}(1) = |\varphi(y)| + e^{-i\arg(\varphi(y))}\frac{\sin(\cos^{-1}(|\varphi(y)|)}{(1-|\varphi(y)|^2)^{1/2}}(y - \varphi(y)) =$$

$$= |\varphi(y)| \cdot 1 + e^{-i\arg(\varphi(y))}(y - \varphi(y)) = e^{-i\arg(\varphi(y))} \cdot y.$$

□

Theorem 3.3. *The actions of $\mathcal{U}_\varphi(\mathcal{A})$ on $\mathbb{S}_\varphi$ and $\mathbb{P}_\varphi$ are transitive.*

Proof. It suffices to prove the first claim. Let $y \in \mathbb{S}_\varphi$. If $\varphi(y) \neq 0$ and $y \notin \mathbb{C} \cdot 1$, by Lemma 3.2, there exists $z \in \mathcal{A}$ such that

$$e^{i(z\otimes 1+1\otimes z)}(1) = e^{-i\theta} \cdot y,$$

i.e. $y = e^{i\theta}e^{i(z\otimes 1+1\otimes z)}(1) = e^{i(z\otimes 1+1\otimes z+\arg(\varphi(y))I)}(1)$, with

$$e^{i(z\otimes 1+1\otimes z+\arg(\varphi(y))I)} \in \mathcal{U}_\varphi(\mathcal{A}).$$

If $\varphi(y) = 0$, by the Remark above,

$$e^{i\frac{\pi}{2}(1\otimes y+y\otimes 1)}(1) = \cos\left(\frac{\pi}{2}\varphi(y^\star y)^{1/2}\right) \cdot 1 + i\frac{\sin(\frac{\pi}{2}\varphi(y^\star y)^{1/2})}{\frac{\pi}{2}\varphi(y^\star y)^{1/2}} \cdot \frac{\pi}{2}y = iy,$$

and the proof follows as above. Finally, the case $\varphi(y) \neq 0$ and $y = \lambda \cdot 1$ is trivial. □

Note that $\mathbb{P}_\varphi = exp(\mathcal{F}(\mathcal{A}))$.

In the above result it was shown that the invertibles in $\mathcal{U}_\varphi(\mathcal{A})$ linking 1 to y are exponentials. Thus

Corollary 3.4. *The sphere $\mathbb{S}_\varphi$ and the projective space $\mathbb{P}_\varphi$ are connected.*

4 Differentiable structure of $\mathbb{S}_\varphi$ and $\mathbb{P}_\varphi$

We shall see that $\mathbb{S}_\varphi$ is a C^∞ complemented submanifold of $\mathcal{A}$, and that $\mathbb{P}_\varphi$ is a C^∞ differentiable manifold, presenting both spaces as homogeneous spaces of $\mathcal{U}_\varphi(\mathcal{A})$.

To prove the assertion on $\mathbb{S}_\varphi$, we shall use again Lemma 2.6:

Proposition 4.1. *The sphere $\mathbb{S}_\varphi$ is a C^∞ complemented submanifold of $\mathcal{A}$, and a homogeneous space of $\mathcal{U}_\varphi(\mathcal{A})$. For any fixed $x_0 \in \mathbb{S}_\varphi$, the map*

$$\pi_{x_0} : \mathcal{U}_\varphi(\mathcal{A}) \to \mathbb{S}_\varphi, \quad \pi_{x_0}(U) = U(x_0)$$

is a C^∞-submersion.

Proof. First, recall from Corollary 3.4 that π_{x_0} is onto. In the frame of Lemma 2.6, consider $\mathbb{S}_\varphi \subset \mathcal{A}$ with the relative topology, and fix $x_0 \in \mathbb{S}_\varphi$. To prove that π_{x_0} is open, we exhibit a continuous local cross section near x_0 (local cross sections near other points of $\mathbb{S}_\varphi$ are obtained by translating this one with the group action). By Proposition 2.5, there exists r_{x_0} such that if $P \in \mathcal{P}_{\mathbf{a}}$ satisfies

$$\|P - x_0 \otimes x_0\|_{\mathbf{a}} < r_{x_0},$$

then there exists $V_P \in \mathcal{U}_\varphi(\mathcal{A})$, which is a smooth function of P, such that

$$V_P(x_0 \otimes x_0)V_P^\sharp = V_P(x_0) \otimes V_P(x_0) = P$$

and $V_{x_0\otimes x_0} = 1$. Note that if $a, b \in \mathcal{A}$, then (by the Cauchy-Schwarz inequality)

$$\|a \otimes b\| = \sup_{\|x\|_\infty \le 1} \|\varphi(b^\star x)a\|_\infty = \|a\|_\infty \sup_{\|x\|_\infty \le 1} |\varphi(b^\star x)| \le \|a\|_\infty \varphi(b^\star b)^{1/2} \sup_{\|x\|_\infty \le 1} \varphi(x^\star x)^{1/2}$$

$$= \|a\|_\infty \|b\|_\varphi \|\varphi\| = \|a\|_\infty \|b\|_\varphi \le \|a\|_\infty \|b\|_\infty.$$

Thus in particular

$$\|a \otimes b\|_{\mathbf{a}} = \max\{\|a \otimes b\|, \|(a \otimes b)^\sharp\|\} = \max\{\|a \otimes b\|, \|b \otimes a\|\} \le \|a\|_\infty \|b\|_\infty.$$

(More precisely: $\|a \otimes b\|_{\mathbf{a}} \le \max\{\|a\|_\infty \|b\|_\varphi, \|a\|_\varphi \|b\|_\infty\}$)

If $y \in \mathbb{S}_\varphi$ satisfies that $\|y - x_0\| < \frac{r_{x_0}}{2\|x_0\|_\infty + 1}$, then

$$\|y \otimes y - x_0 \otimes x_0\|_{\mathbf{a}} \le \|y \otimes y - y \otimes x_0\|_{\mathbf{a}} + \|y \otimes x_0 - x_0 \otimes x_0\|_{\mathbf{a}} = \|y \otimes (y - x_0)\|_{\mathbf{a}} + \|(y - x_0) \otimes x_0\|_{\mathbf{a}}$$

$$\le \|y\|_\infty \|y - x_0\|_\infty + \|y - x_0\|_\infty \|x_0\|_\infty.$$

Note that since $\frac{r_{x_0}}{2\|x_0\|_\infty + 1} \le 1$, one has that for such y

$$\|y\|_\infty < \|y - x_0\|_\infty + \|x_0\|_\infty \le 1 + \|x_0\|_\infty.$$

Then

$$\|y \otimes y - x_0 \otimes x_0\|_{\mathbf{a}} < \|y - x_0\|_\infty (1 + 2\|x_0\|_\infty) < r_{x_0}.$$

Then by Proposition 2.5, there exists $V_y \in \mathcal{U}_\varphi(\mathcal{A})$, depending continuously on $y \otimes y$ (and therefore on y) such that

$$V_y^\sharp(y) \otimes V_y^\sharp(y) = V_y^\sharp(y \otimes y) V_y = x_0 \otimes x_0,$$

i.e. $y' = V_y^\sharp(y)$ satisfies $y' \otimes y' = x_0 \otimes x_0$. Put $U_y = x_0 \otimes y' + 1 - x_0 \otimes x_0$. Note that $U_y \in \mathcal{U}_\varphi(\mathcal{A})$. Indeed, since $y' \otimes x_0(1 - x_0 \otimes x_0) = (1 - x_0 \otimes x_0) y' \otimes x_0 = 0$,

$$U_y^\sharp U_y = (y' \otimes x_0 + 1 - x_0 \otimes x_0)(x_0 \otimes y' + 1 - x_0 \otimes x_0) = (y' \otimes x_0)(x_0 \otimes y') + 1 - x_0 \otimes x_0$$

$$= x_0 \otimes x_0 + 1 - x_0 \otimes x_0 = 1.$$

and similarly $U_y U_y^\sharp = 1$. It is clear that also U_y depends continuously on y. Moreover

$$U_y^\sharp(x_0) = y' \otimes x_0(x_0) = y'.$$

Then μ_{x_0} defined on y such that $\|y - x_0\|_\infty < \frac{r_{x_0}}{2\|x_0\|_\infty + 1}$ by

$$\mu_{x_0}(y) = V_y U_y^\sharp \in \mathcal{U}_\varphi(\mathcal{A})$$

satisfies $[\mu_{x_0}(y)](x_0) = V_y U_y^\sharp(x_0) = V_y(y') = y$, i.e. is a continuous local cross section for π_{x_0} defined near x_0. Thus π_{x_0} is open.

The differential of π_{x_0} at $1 \in \mathcal{U}_\varphi(\mathcal{A})$ (regarded as a map with values in $\mathcal{A}$) is

$$\delta_{x_0} = d(\pi_{x_0})_1 : \mathcal{B}_{as}(\mathcal{A}) \to \mathcal{A}, \ \delta_{x_0}(A) = A(x_0).$$

Note that $A \in \mathcal{B}_{as}(\mathcal{A})$ satisfies $A(x_0) = 0$ if and only if $A(x_0) \otimes x_0 = -x_0 \otimes A(x_0) = 0$, i.e. the matrix of A in terms of the projection $x_0 \otimes x_0$ has only the $2, 2$-entry that is non zero. Thus a supplement for $N(\delta_{x_0})$ in $\mathcal{B}_{as}(\mathcal{A})$ is the space of matrices in terms of $x_0 \otimes x_0$ which have trivial $2, 2$- entry, i.e. the elements B of $\mathcal{B}_{as}(\mathcal{A})$ such that $(1 - x_0 \otimes x_0) B (1 - x_0 \otimes x_0) = 0$.

The range $R(\delta_{x_0})$ of this linear map is

$$R(\delta_{x_0}) = \{y \in \mathcal{A} : Re(\varphi(x_0^\star y)) = 0\}.$$

Indeed, if $y = A(x_0)$ (with $A^\sharp = -A$), then

$$\varphi(x_0^\star y) = \varphi(x_0^\star A(x_0)) = -\varphi(A(x_0)^\star x_0) = -\varphi(y^\star x_0) = -\overline{\varphi(x_0^\star y)}.$$

Conversely, if $Re(\varphi(x_0^* y)) = 0$, consider $A = y \otimes x_0 - x_0 \otimes y - \varphi(x_0^* y) x_0 \otimes x_0$. Clearly $A \in \mathcal{B}_{as}(\mathcal{A})$, and

$$A(x_0) = y - \varphi(x_0^* y)x_0 - \varphi(y^* x_0)x_0 = y.$$

In particular, $R(\delta_{x_0})$ is the nullspace of a bounded real functional in $\mathcal{A}$, thus it is a (real) complemented subspace of $\mathcal{A}$. Thus Lemma 2.6 applies and the proof follows. □

Remark 4.2. In the above proof, one obtains that the tangent space at a given $x_0 \in \mathbb{S}_\varphi$ is

$$(T\mathbb{S}_\varphi)_{x_0} = \{y \in \mathcal{A} : Re\ \varphi(x_0^* y) = 0\}.$$

Indeed, suppose that $x(t)$, $t \in (-r, r)$ is a smooth curve in $\mathbb{S}_\varphi$ with $x(0) = x_0$ and $\dot{x}(0) = y$. Since $\varphi(x^*(t)x(t)) = 1$, then

$$0 = \varphi(\dot{x}^*(t)x(t)) + \varphi(x^*(t)\dot{x}(t)),$$

and at $t = 0$,

$$0 = \varphi(y^* x_0) + \varphi(x_0^* y) = 2Re\ \varphi(x_0^* y).$$

Conversely, if $y \in \mathcal{A}$ satisfies $Re\ \varphi(x_0^* y) = 0$, then there exists $A \in \mathcal{B}_{as}(\mathcal{A})$ such that $A(x_0) = y$. In term matrix (based on $x_0 \otimes x_0$):

$$A = \begin{pmatrix} \lambda i x_0 & -(x_0 \otimes z) \\ z \otimes x_0 & 0 \end{pmatrix},$$

where $z = y - \varphi(x_0^* y)x_0$ and $\lambda i = \varphi(x_0^* y)$ with $\lambda \in \mathbb{R}$. Thus

$$x(t) = e^{tA}(x_0) \in \mathbb{S}_\varphi$$

satisfies $x(0) = x_0$ and $\dot{x}(0) = A(x_0) = y$, i.e. $y \in (T\mathbb{S}_\varphi)_{x_0}$.

Remark 4.3. The facts that $\mathbb{P}_\varphi$ is a differentiable manifold with the quotient topology, and that the map $\pi_{[x_0]}$ is a submersion for any $[x_0] \in \mathbb{P}_\varphi$ are easier to prove. Note that here we do not claim that $\mathbb{P}_\varphi$ is a submanifold (it does not lie in a Banach space). Indeed, in the quotient topology, $\mathbb{P}_\varphi$ identifies (is homeomorphic) with $\mathcal{P}_1(\mathcal{A}, \varphi)$, the manifold of *rank one* projections of $\mathcal{B}_{\mathbf{a}}(\mathcal{A})$, which coincides with the $\mathcal{U}_\varphi(\mathcal{A})$-orbit of $x_0 \otimes x_0$, as seen before. By the results earlier in this section, this space is a differentiable manifold and the projection map a submersion.

4.1 The selfadjoint part of $\mathbb{S}_\varphi$

By Remark 4.2, the tangent space of $\mathbb{S}_\varphi$ at 1 is $\{x \in \mathcal{A} : Re(\varphi(x)) = 0\}$.

Lemma 4.4. *The tangent space of $\mathbb{S}_\varphi$ at 1 naturally decomposes as the (real linear) direct sum*

$$(T\mathbb{S}_\varphi)_1 = \mathcal{A}_{ah} \oplus N(\varphi)_s,$$

where $\mathcal{A}_{ah}$ denotes the space of antihermitian elements of $\mathcal{A}$ and $N(\varphi)_s = N(\varphi) \cap \mathcal{A}_s$ the selfadjoint elements in the nullspace of φ.

Proof. Clearly if $x \in (T\mathbb{S}_\varphi)_1$, then also $x^* \in (T\mathbb{S}_\varphi)_1$. Thus $(T\mathbb{S}_\varphi)_1 = ((T\mathbb{S}_\varphi)_1 \cap A_{ah}) \oplus ((T\mathbb{S}_\varphi)_1 \cap \mathcal{A}_s)$.

If $y \in A_{ah}$, then $0 = \varphi(y^*) + \varphi(y) = 2Re(\varphi(y))$ and thus $(T\mathbb{S}_\varphi)_1 \cap A_{ah} = A_{ah}$.

On the other hand, if $y \in (T\mathbb{S}_\varphi)_1 \cap \mathcal{A}_s$, then $\varphi(y) = \varphi(y^*) = \overline{\varphi(y)}$ and thus $\varphi(y) = 0$. Hence $(T\mathbb{S}_\varphi)_1 \cap \mathcal{A}_s = N(\varphi)_s$. □

Consider the map

$$\mu : (T\mathbb{S}_\varphi)_1 = \mathcal{A}_{ah} \oplus N(\varphi)_s \to \mathbb{S}_\varphi, \quad \mu(a, b) = e^a(e^{b\otimes 1 - 1\otimes b}(1)).$$

Note that $b \otimes 1 - 1 \otimes b$ belongs to $\mathcal{F}_{as}(\mathcal{A})$ and thus $e^{b\otimes 1-1\otimes b} \in \mathcal{U}_\varphi(\mathcal{A})$ and $x_b = e^{b\otimes 1-1\otimes b}(1) \in \mathbb{S}_\varphi$. Then

$$\varphi(\mu^*(a, b)\mu(a, b)) = \varphi(x_b^* x_b) = 1.$$

It is clear that μ is C^∞. By elementary computations similar to that of Remark 3.1, if $b \neq 0$ one has

$$x_b = \cos(\varphi(b^2)^{1/2})1 + \frac{1}{\varphi(b^2)^{1/2}}\sin(\varphi(b^2)^{1/2})b = \cos(\varphi(b^2)^{1/2})1 + \operatorname{sinc}(\varphi(b^2)^{1/2})b\ ,$$

which makes sense even if $b = 0$. Note that $e^a \in \mathcal{U}_\mathcal{A}$ and that x_b is a selfadjoint element (in $\mathbb{S}_\varphi$).

Let $a(t)$, $b(t)$ be smooth curves in $\mathcal{A}_{ah}$ and $N(\varphi)_s$ with $a(0) = b(0) = 0$, $\dot{a}(0) = z$ and $\dot{b}(0) = y$, where $x = z + y$ is an arbitrary element of $(T\mathbb{S}_\varphi)_1$. Then it is clear that (using that the differential of the exponential map at the origin is the identity map)

$$d\mu_0(x) = \frac{d}{dt}\mu(a(t), b(t))|_{t=0} = \dot{a}(0) + (\dot{b}(0) \otimes 1 - 1 \otimes \dot{b}(0))(1) = z + y = x$$

because $(\dot{b}(0) \otimes 1)(1) = \varphi(y)1 = 0$. Therefore, using the inverse function theorem, one has the following result.

Proposition 4.5. *There exist balls $B_{r_{as}}$ and B_{r_s} of radius r_{as} and r_s around the origin in $\mathcal{A}_{ah}$ and $N(\varphi)_s$ respectively, and an open set $\mathcal{V}$ in $\mathbb{S}_\varphi$ containing 1 such that*

$$\mu : B_{r_{as}} \oplus B_{r_s} \to \mathcal{V}$$

is a C^∞ diffeomorphism. In particular, any element x in $\mathcal{V}$ factorizes

$$x = ux_b$$

with u unitary and x_b a selfadjoint element in $\mathbb{S}_\varphi$. The factorization is unique if one requires that u and x_b belong to the exponential of $B_{r_{as}}$ and B_{r_s}.

Remark 4.6. Let $x \in \mathbb{S}_\varphi$. Then $|x| \in \mathbb{S}_\varphi$. Indeed,

$$\varphi(|x|^*|x|) = \varphi(|x|^2) = \varphi(x^*x) = 1.$$

However, if $x = v|x|$ is the polar decomposition of x (supposing that this decomposition done in $\mathcal{B}(\mathcal{L})$ remains in $\mathcal{A}$, i.e. $v \in \mathcal{A}$), then $v \in \mathbb{S}_\varphi$ if and only if v is an isometry: since $\|v\| = 1$ (and φ is faithful), $\varphi(v^*v) = 1$ is equivalent to $v^*v = 1$. This in turn means that $N(x)$ is trivial in $\mathcal{L}$. For instance, if $x \in \mathbb{S}_\varphi$ is invertible, then $v \in \mathcal{A}$ and thus $v \in \mathbb{S}_\varphi$. The polar decomposition differs from the local factorization above: x_b above may not be positive.

The unitary group $\mathcal{U}_\mathcal{A}$ clearly is a submanifold of $\mathbb{S}_\varphi$. The same holds for the selfadjoint part $\mathbb{S}_{\varphi,s}$ of $\mathbb{S}_\varphi$,

$$\mathbb{S}_{\varphi,s} = \{x \in \mathcal{A}_s : \varphi(x^2) = 1\} = \mathbb{S}_\varphi \cap \mathcal{A}_s.$$

Proposition 4.7. *$\mathbb{S}_{\varphi,s}$ is a submanifold of $\mathcal{A}$, and therefore also of $\mathbb{S}_\varphi$.*

Proof. Consider the C^∞ map

$$q : \mathcal{A}_s \to \mathbb{R}_{>0},\ \ q(a) = \varphi(a^2).$$

It is clearly a retraction: $s : \mathbb{R}_{>0} \to \mathcal{A}_s$, $s(t) = t^{1/2} \cdot 1$ is a smooth cross section for q. In particular, q is a submersion. Then

$$\mathbb{S}_{\varphi,s} = q^{-1}(\{1\})$$

is a submanifold of $\mathcal{A}_s$. □

Let us examine the properties of the restriction μ_s of the map μ above,

$$\mu_s : N(\varphi)_s \to \mathbb{S}_{\varphi,s} \quad , \quad \mu_s(a) = e^{a\otimes 1 - 1\otimes a}(1) = \cos(\varphi(a^2)^{1/2}) \cdot 1 + \frac{\sin(\varphi(a^2)^{1/2})}{\varphi(a^2)^{1/2}} \cdot a$$

Let $x \in \mathbb{S}_{\varphi,s}$ and $a \in N(\varphi)$ so that $\mu(a) = x$. Since $\varphi(\mu(a)) = \cos(\varphi(a^2)^{1/2}) = \varphi(x) \in [-1, 1]$, then $\varphi(a^2)^{1/2} = \cos^{-1}(\varphi(x)) + 2n\pi$ or $-\cos^{-1}(\varphi(x)) + 2n\pi$. By elementary computations,

$$\mu_s(a) = \varphi(x) \cdot 1 + \frac{\sin(\varphi(a^2)^{1/2})}{\varphi(a^2)^{1/2}} \cdot a = x$$

$$\frac{\sin(\varphi(a^2)^{1/2})}{\varphi(a^2)^{1/2}} \cdot a = x - \varphi(x)$$

It follows that if $|\varphi(x)| \neq 1$,

$$\mu_s^{-1}(x) = \left\{ a \in N(\varphi) : a = (x - \varphi(x)) \frac{\cos^{-1}(\varphi(x)) + 2n\pi}{\left(1 - \varphi(x)^2\right)^{1/2}}, n \in \mathbb{Z} \right\}$$

and, since $|\varphi(x)| = 1$ implies $x = 1$ or $x = -1$, so

$$\mu_s^{-1}(1) = \left\{ a \in N(\varphi) : \varphi(a^2)^{1/2} = 2n\pi \right\}$$

$$\mu_s^{-1}(-1) = \left\{ a \in N(\varphi) : \varphi(a^2)^{1/2} = (2n+1)\pi \right\}.$$

These are the fibers of the map μ_s over each $x \in \mathbb{S}_{\varphi,s}$.

Proposition 4.8. *The map μ_s is onto. It has a smooth right inverse defined on $\mathbb{S}_{\varphi,s} \setminus \{1, -1\}$. Moreover it is a covering space.*

Proof. The right inverse is given by

$$\theta(x) = \frac{\cos^{-1}(\varphi(x))}{(1 - \varphi(x)^2)^{1/2}} .(x - \varphi(x)).$$

Note that since $\varphi(x^2) = 1$, $|\varphi(x)| \leq 1$. Moreover $\varphi(x)^2 = 1$ only if $x = 1, -1$: in the Cauchy-Schwarz inequality

$$1 = |\varphi(x)| \leq \varphi(x^2)^{1/2}\varphi(1)^{1/2} = 1$$

one has equality, thus $x = \lambda 1$, with $\lambda^2 = 1$. Thus θ is well defined and smooth in $\mathbb{S}_{\varphi,s} \setminus \{1, -1\}$. The fact that $\mu(\theta(x)) = x$ is an elementary computation.

In order to prove that μ_s is onto, it suffices to find selfadjoint elements a_1, a_2 with $\varphi(a_i) = 0$ such that $\mu(a_1) = 1$ and $\mu(a_2) = -1$. Put $a_1 = 0$. Take $b^* = b$ such that $b \neq \lambda 1$ and $\varphi(b) \neq 0$. Let $b' = b - \varphi(b) \cdot 1$ and then $\varphi(b') = 0$. Suppose that $\varphi(b'^2) \neq 0$. Thus $a_2 = \frac{\pi}{\varphi(b'^2)^{1/2}} b'$ verifies $\mu(a_2) = -1$.

Let us prove that $\mu_s : N(\varphi)_s \to \mathbb{S}_{\varphi,s}$ is a covering space. We have to show that every point in $\mathbb{S}_{\varphi,s}$ has a neighborhood V such that $\mu^{-1}(V) = \bigcup U_\alpha$ where U_α are disjoint open subsets of $N(\varphi)_s$ and $\mu|_{U_\alpha}$ is a homeomorphism of U_α onto V. It is clearly that this is verified in $x \neq 1$ or $x \neq -1$.

Suppose that $a(t)$, $t \in (-r, r)$ is a smooth curves in $N(\varphi)_s$ with $a(0) = 0$ and $\dot{a}(0) = v$, where $v \in N(\varphi)_s$. Then

$$D\mu_s(v) = \dot{\mu}_s(a(t))|_{t=0} = (\dot{a}(0) \otimes 1 - 1 \otimes \dot{a}(0))(1) = v.$$

Using the inverse function theorem, there exist a ball B_r of radius r around the origin in $N(\varphi)$ and an open set $\mathcal{V}$ in $\mathbb{S}_{\varphi,s}$ containing 1 such that

$$\mu_s : B_r \to \mathcal{V}$$

is a C^∞ diffeomorphism. Note that this result is a particular case of proposition 4.5.

Analogously, if $a(t)$, $t \in (-r, r)$ is a smooth curve in $N(\varphi)_s$ with $a(0) = a_2$ and $\dot{a}(0) = v$, where $v \in (TN(\varphi)_s)_{a_2}$ and a_2 verifies $\mu(a_2) = -1$. By elementary computations similar as Remark 3.1, one has (in terms of $1 \otimes 1$)

$$e^{a_2 \otimes 1 - 1 \otimes a_2} = \begin{pmatrix} \cos(\varphi(a_2^2)^{1/2}) & -(1 \otimes a_2)\mathrm{sinc}(\varphi(a_2^2)^{1/2}) \\ -(a_2 \otimes 1)\mathrm{sinc}(\varphi(a_2^2)^{1/2}) & \cos(\varphi(a_2^2)^{1/2}) \end{pmatrix} = -Id_{T(N(\varphi)_s)_{a_2}}$$

and then

$$D\mu_s(v) = \dot{\mu}_s(a(t))|_{t=0} = -(\dot{a}(0) \otimes 1 - 1 \otimes \dot{a}(0))(1) = -v$$

Therefore, using the inverse function theorem, there exists a ball B_r of radius r around the a_2 in $N(\varphi)_s$ and an open set $\mathcal{V}'$ in $\mathbb{S}_{\varphi,s}$ containing -1 such that

$$\mu_s : B_{r'} \to \mathcal{V}'$$

is a C^∞ diffeomorphism.

In both cases, there exists a ball V in $\mathbb{S}_{\varphi,s}$ such that

$$B_r = \mu_s^{-1}(V) = \bigcup_{n \in \mathbb{Z}} \left\{ a \in N(\varphi) : \varphi(a^2)^{1/2} = \cos^{-1}(\varphi(x)) + 2n\pi, x \in \mathcal{V} \right\}.$$

□

5 A pre-Hilbert-Riemann metric for $\mathbb{P}_\varphi$

As written in Remark 4.3, we shall identify $\mathbb{P}_\varphi \simeq \mathcal{P}_1(\mathcal{A}, \varphi)$. Therefore the tangent space of $\mathbb{P}_\varphi$ at $[x]$ identifies with

$$(T\mathbb{P}_\varphi)_{[x]} \simeq \{a \otimes x + x \otimes a : Re\varphi(x^* a) = 0\} = (T\mathcal{P}_1(\mathcal{A}, \varphi))_{x \otimes x}.$$

Also $\mathcal{P}_1(\mathcal{A}, \varphi) \subset \mathcal{P}_1(\mathcal{L})$, and this last manifold has a well behaved Hilbert-Riemann structure induced by the Frobenius norm (it shall be recalled in Section 5).

We have pointed out though that $\mathcal{P}_1(\mathcal{A}, \varphi)$ is not a submanifold of $\mathcal{P}_1(\mathcal{L})$, the differentiable structure of both spaces is quite different (the inclusion is dense in the topology of $\mathcal{B}(\mathcal{L})$). Nevertheless this inclusion has the remarkable property to be locally *geodesically* complete: if two elements in $\mathcal{P}_1(\mathcal{A}, \varphi)$ lie close, the minimal geodesic of $\mathcal{P}_1(\mathcal{L})$ which joins them lies inside $\mathcal{P}_1(\mathcal{A}, \varphi)$. This property would suggest to consider in $\mathcal{P}_1(\mathcal{A}, \varphi)$ the metric induced by this inclusion. We shall present below an intrinsic metric in $\mathbb{P}_\varphi$, and will show thereafter that it is (a multiple) of the metric induced by $\mathcal{P}_1(\mathcal{L})$.

Let us first characterize the tangent spaces of $\mathbb{P}_\varphi$ as quotient spaces.

Lemma 5.1. *Let $[x_0] \in \mathbb{P}_\varphi$. Then $(T\mathbb{P}_\varphi)_{[x_0]}$ is naturally isomorphic to*

$$\{a \in \mathcal{A} : \varphi(x_0^* a) \in i\mathbb{R}\}/i\mathbb{R} \cdot x_0, \tag{2}$$

i.e. a, b define the same tangent vector at $[x_0]$ if $\varphi(a^* x_0), \varphi(b^* x_0) \in i\mathbb{R}$ and $a - b = irx_0$ for some $r \in \mathbb{R}$.

By naturally isomorphic we mean the following: if one chooses another representative x_0' for $[x_0]$, i.e. $x_0' = wx_0$ for some $w \in \mathbb{T}$, then the mapping

$$a' \mapsto \bar{w}a'$$

sends $\{a' \in \mathcal{A} : \varphi(x_0'^* a') \in i\mathbb{R}\}$ onto $\{a \in \mathcal{A} : \varphi(x_0^* a) \in i\mathbb{R}\}$ and $i\mathbb{R} \cdot x_0'$ onto $i\mathbb{R} \cdot x_0$, and thus defines an isomorphism between the quotients.

Proof. Suppose that $x(t)$, $t \in (-r, r)$ is a smooth curve in $\mathbb{S}_\varphi$ with $x(0) = x_0$ and $\dot{x}(0) = a$. By Remark 4.2, we know that $Re\ \varphi(x_0^* a) = 0$. Let $y(t) = w(t)x(t)$ be another smooth curve in $\mathbb{S}_\varphi$ equivalent to $x(t)$, i.e $w(t) \in \mathbb{T}$ (we may suppose $w(0) = 1$ without loss of generality). Put $b = \dot{y}(0)$. Then differentiating at $t = 0$ one obtains

$$b = \dot{w}(0)x_0 + a.$$

Note that $\dot{w}(0) \in i\mathbb{R}$. Then it is clear that $b - a \in i\mathbb{R} \cdot x_0$. It follows that the tangent space is contained in the quotient (2).

Let us prove that any element in this quotient can be realized as a velocity vector. Pick $a \in \mathcal{A}$ with $\varphi(x_0^* a) \in i\mathbb{R}$. Note that $a - \varphi(x_0 a^*) \in (T\mathbb{S}_\varphi)_{x_0}$ and $[a - \varphi(x_0 a^*)]$ is the same as the class of a in the quotient (2). Again, using Remark 4.2, if $A = a \otimes x_0 - x_0 \otimes a$ in $\mathcal{B}_{as}(\mathcal{A})$ then $A(x_0) = a - \varphi(x_0 a^*)$ and

$$e^{t(a \otimes x_0 - x_0 \otimes a)} \in \mathcal{U}_\varphi(\mathcal{A})$$

is a smooth curve. Thus $\gamma(t) = [e^{t(a\otimes x_0 - x_0\otimes a)}(x_0)]$ is a smooth curve in $\mathbb{P}_\varphi$ with

$$\dot\gamma(0) = [A(x_0)] = [a].$$

□

Let us define a Riemannian metric in $\mathbb{P}_\varphi$:

Definition 5.2. For $[x_0] \in \mathbb{P}_\varphi$ and $[a]$ a tangent vector at $[x_0]$, put

$$|[a]|_{[x_0]} = \inf\{\|a - ir\cdot x_0\|_\varphi : r \in \mathbb{R}\},$$

i.e. the quotient norm in the quotient (2) induced by the norm $\|\cdot\|_\varphi$ in $\mathcal{A}$.

Clearly the metric is well defined (it does not depend on the representative of $[x_0]$).

Note that since $\mathcal{A}$ is not complete with the norm $\|\cdot\|_\varphi$, this quotient norm is non complete in $(T\mathbb{P}_\varphi)_{[x_0]}$. Also note that the orthogonal projection

$$P : \{a \in \mathcal{A} : \varphi(x_0^* a) \in i\mathbb{R}\} \to i\mathbb{R}\cdot x_0$$

is given by the state φ: $P(a) = \varphi(x_0^* a)x_0$. Therefore the infimum at the quotient norm is in fact a minimum, given by

$$|[a]|_{[x_0]} = \|a - \varphi(x_0^* a)x_0\|_\varphi = \{\varphi(a^* a) - |\varphi(x_0^* a)|^2\}^{1/2}.$$

This quantity is positive: $\varphi(a^* a) = |\varphi(x_0^* a)|^2$ means equality in the Cauchy-Schwarz inequality

$$|\varphi(x_0^* a)| \le \varphi(a^* a)^{1/2}\varphi(x_0^* x_0)^{1/2} = \varphi(a^* a)^{1/2},$$

and therefore $a = \lambda x_0$, then $\varphi(x_0^* a) = \lambda \in i\mathbb{R}$, and thus $[a] = 0$.

From these observations, it follows that this metric coincides with (a multiple of) the Frobenius norm in $\mathcal{P}_1(\mathcal{A}, \varphi)$:

Proposition 5.3. *Let $[x] \in \mathbb{P}_\varphi$ and $[a] \in (T\mathbb{P}_\varphi)_{[x]}$. Then*

$$|[a]|_{[x]} = \frac{1}{\sqrt{2}} Tr((a\otimes x + x\otimes a)^2)^{1/2} = \frac{1}{\sqrt{2}}\|a\otimes x + x\otimes a\|_{HS},$$

where Tr denotes the usual trace in $\mathcal{B}(\mathcal{L})$ and $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm.

Proof. Straightforward computations show that

$$(a\otimes x + x\otimes a)^2 = (\varphi(a^* x)x)\otimes a + (\varphi(a^* a)x)\otimes x + (\varphi(x^* x)a)\otimes a + (\varphi(x^* a)a)\otimes x.$$

Thus (using that $Tr(b\otimes c) = \varphi(c^* b)$)

$$Tr((a\otimes x + x\otimes a)^2) = 2\varphi(a^* a) + \varphi(a^* x)^2 + \varphi(x^* a)^2.$$

Note that $\varphi(a^* x) = \overline{\varphi(x^* a)} \in i\mathbb{R}$, and thus

$$\varphi(a^* x)^2 = \varphi(x^* a)^2 = -|\varphi(x^* a)|^2.$$

Therefore

$$Tr((a\otimes x + x\otimes a)^2)^{1/2} = \sqrt{2}\{\varphi(a^* a) - |\varphi(x^* a)|^2\}^{1/2} = \sqrt{2}|[a]|_{[x]}.$$

□

6 Minimality of geodesics in $\mathbb{P}_\varphi$

The results in this section refer to minimal curves in $\mathbb{P}_\varphi$, i.e. curves $\delta(t) = [\alpha(t)]$, $\alpha(t) \in \mathbb{S}_\varphi$, with minimal length joining fixed endpoints. The length of the curve $\delta(t)$, $0 \le t \le 1$, is given by

$$\ell(\delta) = \int_0^1 |[\dot{\delta}(t)]|_{\delta(t)} dt$$

where $|[v]|_{[\delta(t)]}$ denotes the quotient norm as in the Definition 5.2.

Note that if $P_0 \in \mathcal{P}_{\mathbf{a}}$ and $Z \in \mathcal{B}_{as}(\mathcal{A})$, then the geodesic $\delta(t) = e^{tZ} P_0 e^{-tZ}$ remains inside $\mathcal{P}_{\mathbf{a}}$. We shall call these curves geodesics in $\mathcal{P}_{\mathbf{a}}$, though we have not defined a linear connection in $\mathcal{P}_{\mathbf{a}}$. In the case when n (the rank of P_0) equals 1, we shall call them *geodesics of* $\mathbb{P}_\varphi$. Note also that if $\delta(t) = e^{tZ} P_0 e^{-tZ}$ is a geodesic of $\mathcal{P}_{\mathbf{a}}$ and $U \in \mathcal{U}_\varphi(\mathcal{A})$, then $U\delta(t)U^\sharp$ is also a geodesic, with exponent $UZU^\sharp$. If $n = 1$, this means that $[Ue^{tZ}(x)]$ is a geodesic of $\mathbb{P}_\varphi$.

Elements P in $\mathcal{P}_{\mathbf{a}}$ extend to orthogonal projections $\bar{P}$ in $\mathcal{L}$. Conversely, any orthogonal projection E in $\mathcal{L}$ which leaves $\mathcal{A} \subset \mathcal{L}$ invariant, i.e. $E(\mathcal{A}) \subset \mathcal{A}$, induces an element $E|_{\mathcal{A}}$ in $\mathcal{P}_{\mathbf{a}}$. Also, $\mathcal{P}_1(\mathcal{L}) \subset Gr(\mathcal{L})$ the Grassmann manifold of $\mathcal{L}$ which is just the set of orthogonal projections of $\mathcal{L}$. In [3, 4] a reductive structure was introduced in the Grassmann manifold of a Hilbert space (parametrized by selfadjoint projections). In [9] the geometry of the restricted or Sato Grassmannian was studied. Later in [10] it was characterised when there are unique geodesics between projections. Let us summarize this information, in the case of $\mathcal{L}$, in the following remark.

Remark 6.1.

1. The space $\mathcal{P}(\mathcal{L})$ is a homogeneous space under the action of the unitary group $\mathcal{U}(\mathcal{L})$ by inner conjugation. The orbits of the action coincide with the connected components of $\mathcal{P}(\mathcal{L})$. In particular, the projections of rank one form a connected component and coincide the space $P_1(\mathcal{L})$.
2. There is a natural linear connection in $\mathcal{P}(\mathcal{L})$. If $dim(\mathcal{L}) < \infty$, it is the Levi-Civita connection of the Riemannian metric which consists of considering the Frobenius inner product at every tangent space. It is based on the diagonal - codiagonal decomposition of $\mathcal{B}(\mathcal{L})$. To be more specific, given $P_0 \in \mathcal{P}(\mathcal{L})$, the tangent space of $\mathcal{P}(\mathcal{L})$ at P_0 consists of all selfadjoint codiagonal matrices (in terms of P_0). The linear connection in $\mathcal{P}(\mathcal{L})$ is induced by a reductive structure, where the horizontal elements at P_0 (in the Lie algebra of $\mathcal{U}(\mathcal{L})$: the space of antihermitian elements of $\mathcal{B}(\mathcal{L})$) are the codiagonal antihermitian operators. The geodesics of $\mathcal{P}(\mathcal{L})$ which start at P_0 are curves of the form
$$\delta(t) = e^{tZ} P_0 e^{-tZ}, \tag{3}$$
with $Z^* = -Z$, codiagonal with respect to P_0.
3. It was proved in [4] that if $P_0, P_1 \in \mathcal{P}(\mathcal{L})$ satisfy $\|P_0 - P_1\| < 1$, then there exists a unique geodesic (up to reparametrization) joining P_0 and P_1. This condition is not necessary for the existence of a unique geodesic.
4. In [10] a necessary and sufficient condition was found, in order that there exists a unique geodesic joining two projections P and Q. This is the case if and only if
$$R(P) \cap N(Q) = N(P) \cap R(Q) = \{0\}.$$
5. If $dim(\mathcal{L}) = \infty$, and one endows each tangent space of $\mathcal{P}(\mathcal{L})$ with the usual norm of $\mathcal{B}(\mathcal{L})$, one obtains a continuous (non regular) Finsler metric. In [4] it was shown that the geodesics (3) remain minimal among their endpoints for all t such that
$$|t| \le \frac{\pi}{2\|Z\|}.$$
If $dim(\mathcal{L}) < \infty$, the Frobenius metric is available to measure lengths of curves. In [9] it was shown that the geodesics remain minimal in the Frobenius norm as long as $|t| \le \frac{\pi}{2\|Z\|}$, which is a condition on the usual operator norm.

6. It is sometimes useful to parametrize projections using symmetries S ($S^* = S$, $S^2 = 1$), via the affine map
$$P \longleftrightarrow S_P = 2P - 1.$$
Some algebraic computations are simpler with symmetries. For instance, the condition that the exponent Z (of the geodesic) is P_0-codiagonal means that Z anti-commutes with S_{P_0}. Thus the geodesic (3), in terms of symmetries, can be expressed
$$S_\delta(t) = e^{itZ} S_{P_0} e^{-itZ} = e^{2itZ} S_{P_0} = S_{P_0} e^{-2itZ}.$$

Let us apply the results of the above Remark to the projective space. Note that the Riemannian metric we introduced in $\mathbb{P}_\varphi$ is ($\frac{1}{\sqrt{2}}$-times) the metric of $\mathcal{P}_1(\mathcal{L})$. Therefore

Theorem 6.2. *Let $[x] \in \mathbb{P}_\varphi$ and $[v] \in (T\mathbb{P}_\varphi)_{[x]}$. The unique geodesic $\delta : [0,1] \to \mathbb{P}_\varphi$ which satisfies $\delta(0) = [x]$ and $\dot\delta(0) = [v]$ is given by*
$$\delta(t) = [e^{tv}(x_0)]$$
which is minimal for $|[v]|_{[x]} \leq \frac{\pi}{2\sqrt{2}}$.

Proof. Let $P_0 = x \otimes x \in \mathcal{P}_1(\mathcal{L})$. Since $[v] \in (T\mathbb{P}_\varphi)_{[x]}$, if one chooses $v \in T\mathbb{S}_\varphi$ (representative for $[v]$) then $V = v \otimes x + x \otimes v \in (T\mathcal{P}_1(\mathcal{L}))_{P_0}$ and in matrix terms (based on P_0)
$$V = \begin{pmatrix} 0 & b^* \\ b & 0 \end{pmatrix}$$
where $b = v \otimes x - \varphi(x^* v) x \otimes x$. Consider
$$Z = \begin{pmatrix} 0 & -b^* \\ b & 0 \end{pmatrix}.$$
Clearly, $Z^* = -Z$ and $Z(x) = v - \varphi(x^* y)x$. Then, by Remark 6.1, the curve
$$\delta(t) = e^{tZ} x \otimes x e^{-tZ} = (e^{tZ}x) \otimes (e^{tZ}x), \tag{4}$$
is a geodesic of $\mathcal{P}_1(\mathcal{L})$ and it satisfies $\delta(0) = x \otimes x$ and $\dot\delta(0) = V$. Moreover, if $\|Z\| \leq \frac{\pi}{2}$, the curve is minimal along its path in $Gr(\mathcal{L})$. Note that $Z \in F(\mathcal{A})$ and $Z^\sharp = Z^* = -Z$. Thus $e^{tZ} \in \mathcal{U}_\varphi(\mathcal{A})$ and therefore
$$(e^{tZ}x) \otimes (e^{tZ}x) \subset \mathcal{P}_1(\mathcal{A}).$$
Then $\delta(t)$ is a geodesic of $\mathbb{P}_\varphi$. Since $\sqrt{2}|[v]|_{[x]} = \|V\| \leq \frac{\pi}{2}$, this geodesic is a minimal curve for $t \in [0,1]$. □

Theorem 6.3. *Let $[x], [y] \in \mathbb{P}_\varphi$.*

1. *If $\varphi(y^* x) \neq 0$, then there exists a unique geodesic $\delta(t) = [e^{it(z\otimes 1 + 1\otimes z)}(1)]$ in $\mathbb{P}_\varphi$ which joins $\delta(0) = [x]$ and $\delta(1) = [y]$, which is minimal for $t \in [0,1]$. The element z is given by*
$$z = -ie^{-i\theta} \frac{\cos^{-1}(|\varphi(x^* y)|)}{(2 - 2|\varphi(x^* y)|^2)^{1/2}} (y - \varphi(x^* y)x).$$
The geodesic distance between $[x]$ and $[y]$ is given by
$$d([x],[y]) = \frac{1}{\sqrt{2}} \cos^{-1}(|\varphi(x^* y)|).$$
2. *If $\varphi(x^* y) = 0$, then there exist infinitely many minimal geodesics of $\mathbb{P}_\varphi$ joining $[x]$ and $[y]$. Among them*
$$\delta(t) = [e^{it\frac{\pi}{2}(x\otimes y + y\otimes x)}(x)],$$
whose length is $d([x],[y]) = \frac{\pi}{2\sqrt{2}}$

Proof. First note that the condition $\varphi(y^* x) \neq 0$ does not depend on the representatives x, y. Next, since $\|x\|_\varphi = \|y\|_\varphi = 1$, this condition implies that $|\varphi(y^* x)| < 1$. Suppose first $x = 1$ (and thus $0 < |\varphi(y)| < 1$). As in Lemma 3.2, the element z satisfies

$$[e^{i(z\otimes 1+1\otimes z)}(1)] = [y],$$

or equivalently

$$e^{i(z\otimes 1+1\otimes z)}(1) \otimes e^{-i(z\otimes 1+1\otimes z)}(1) = y \otimes y.$$

We claim further that the curve

$$\delta(t) = e^{it(z\otimes 1+1\otimes z)}(1) \otimes e^{-it(z\otimes 1+1\otimes z)}(1)$$

is a geodesic of $\mathcal{P}_{\mathbf{a}}$, i.e. that the exponent $z \otimes 1 + 1 \otimes z$ is co-diagonal with respect to the projection $1 \otimes 1$. If $x = 1$, z above is given by

$$z = -ie^{-i\theta} \frac{\cos^{-1}(|\varphi(y)|)}{(1 - |\varphi(y)|^2)^{1/2}} \cdot (y - \varphi(y)) = \lambda(y - \varphi(y))$$

where $\theta = arg(\varphi(y))$. Note that $\varphi(z) = 0$. Then

$$(z \otimes 1 + 1 \otimes z)1 \otimes 1 = z \otimes 1 \ \text{ and } \ 1 \otimes 1(z \otimes 1 + 1 \otimes z) = 1 \otimes z,$$

thus

$$z \otimes 1 + 1 \otimes z = (z \otimes 1 + 1 \otimes z)1 \otimes 1 + 1 \otimes 1(z \otimes 1 + 1 \otimes z),$$

which implies that

$$1 \otimes 1(z \otimes 1 + 1 \otimes z)1 \otimes 1 = 0 = (1 - 1 \otimes 1)(z \otimes 1 + 1 \otimes z)(1 - 1 \otimes 1).$$

In the general case, for arbitrary $[x], [y] \in \mathbb{P}_\varphi$, since the action of $\mathcal{U}_\varphi(\mathcal{A})$ is transitive in $\mathbb{P}_\varphi$, there exists $U \in \mathcal{U}_\varphi(\mathcal{A})$ such that $x = U(1)$ and $y = U(y')$. Then

$$\varphi(y') = \varphi(y' 1^*)\varphi(U(y)U(x)^*) = \varphi(yx^*) \neq 0.$$

The element z' which gives the exponent of the geodesic joining $[1]$ and $[y']$ is

$$z' = -ie^{-i\theta'} \frac{\cos^{-1}(|\varphi(y')|)}{(1 - |\varphi(y')|^2)^{1/2}} \cdot (y' - \varphi(y'))$$

with $\theta' = arg(\varphi(y')) = arg(\varphi(x^* y))$. Note that $U(y' - \varphi(y')1) = y - \varphi(x^* y)x$. Thus

$$U(z') = -ie^{-i\theta} \frac{\cos^{-1}(|\varphi(x^* y)|)}{(2 - 2|\varphi(x^* y)|^2)^{1/2}} (y - \varphi(x^* y)x) = z.$$

Therefore

$$\begin{aligned}[U(e^{it(z'\otimes 1+1\otimes z')}(1)] &= [U(e^{it(z'\otimes 1+1\otimes z')}U^\sharp U(1)] = [e^{itU(z'\otimes 1+1\otimes z')U^\sharp} x] \\ &= [e^{it(U(z')\otimes U(1)+U(1)\otimes U(z'))}(x)] = [e^{it(z\otimes x+x\otimes z)}(x)]\end{aligned}$$

is a geodesic joining x and y.

Let us show that δ is minimal. It suffices to consider the case $x = 1$. To prove that δ is minimal in the interval $[0, 1]$, according to Remark 6.1, one needs to show that the usual operator norm $\|z \otimes 1 + 1 \otimes z\|$ of the exponent is less or equal than $\pi/2$. Since $z \otimes 1 + 1 \otimes z$ is $1 \otimes 1$ co-diagonal. Then

$$\|z \otimes 1 + 1 \otimes z\| = \|z \otimes 1\| = \|z\|_\varphi = \cos^{-1}(|\varphi(y)|) \frac{\|y - \varphi(y)\|_\varphi}{(1 - |\varphi(y)|^2)^{1/2}} = \cos^{-1}(|\varphi(y)|) < \pi/2.$$

Lets us compute the geodesic distance $d([x], [y])$, i.e. the length of the geodesic δ. The length is the norm

$$\sqrt{2}|[z]|_{[x]} = \|z - \varphi(z)\|_\varphi = \|z\|_\varphi = \cos^{-1}(|\varphi(x^* y)|) < \pi/2.$$

In order to see that it is unique, consider the projections $x \otimes x$ and $y \otimes y$. Denote by $\mathcal{L}_x$ and $\mathcal{L}_y$ the complex one dimensional spaces which ranges of the extensions of these projections to orthogonal projections in $\mathcal{L}$. It is clear that

$$\mathcal{L}_x \cap \mathcal{L}_y^{\perp} = \{0\} \text{ and } \mathcal{L}_x^{\perp} \cap \mathcal{L}_y = \{0\}$$

because $< x, y >= \varphi(y^* x) \neq 0$. It follows that there exists a unique geodesic in $\mathcal{P}(\mathcal{L})$ joining $x \otimes x$ and $y \otimes y$. The geodesic δ of $\mathcal{P}_{\mathbf{a}}$ extends to a geodesic of $\mathcal{P}(\mathcal{L})$ (and so would any geodesic of $\mathcal{P}_{\mathbf{a}}$). Thus it is unique.

Suppose now that $\varphi(x^* y) = 0$. Again we may suppose $x = 1$. As in Theorem 3.3, one has that $e^{i\frac{\pi}{2}(y \otimes y + y \otimes 1)}(1) = iy$ and thus

$$[e^{i\frac{\pi}{2}(y \otimes 1 + 1 \otimes y)}(1)] = [y].$$

As above, the fact that $\varphi(y) = 0$ implies that $y \otimes 1 + 1 \otimes y$ is $1 \otimes 1$ co-diagonal. Thus δ is a geodesic. Its length is

$$\frac{\pi}{2}\|y\|_{\varphi} = \pi/2.$$

Let $\mathcal{L}_x$ and $\mathcal{L}_y$ the complex lines generated by x and y in $\mathcal{L}$. Since $\mathcal{L}_x \perp \mathcal{L}_y$,

$$\mathcal{L}_x \cap \mathcal{L}_y^{\perp} = \mathcal{L}_x \text{ and } \mathcal{L}_x^{\perp} \cap \mathcal{L}_y = \mathcal{L}_y,$$

and therefore there exist infinitely many geodesics joining $[x]$ and $[y]$. □

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