REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 58, No. 1, 2017, Pages 21–36
Published online: October 27, 2016

QUOTIENT p-SCHATTEN METRICS ON SPHERES

ESTEBAN ANDRUCHOW AND ANDREA C. ANTUNEZ

Abstract. Let $S(H)$ be the unit sphere of a Hilbert space H and $U_p(H)$ the group of unitary operators in H such that $u-1$ belongs to the p-Schatten ideal $B_p(H)$. This group acts smoothly and transitively in $S(H)$ and endows it with a natural Finsler metric induced by the p-norm $\|z\|_p = \mathrm{tr}\left((zz^*)^{p/2}\right)^{1/p}$. This metric is given by

$$\|v\|_{x,p} = \min\{\|z-y\|_p : y \in \mathfrak{g}_x\},$$

where $z \in \mathcal{B}_p(H)_{ah}$ satisfies that $(d\pi_x)_1(z) = z \cdot x = v$ and $\mathfrak{g}_x$ denotes the Lie algebra of the subgroup of unitaries which fix x. We call z a lifting of v. A lifting z_0 is called a minimal lifting if additionally $\|v\|_{x,p} = \|z_0\|_p$. In this paper we show properties of minimal liftings and we treat the problem of finding short curves α such that $\alpha(0) = x$ and $\dot\alpha(0) = v$ with $x \in S(H)$ and $v \in T_xS(H)$ given. Also we consider the problem of finding short curves which join two given endpoints $x, y \in S(H)$.

1. Introduction

Let H be an infinite dimensional Hilbert space and $B(H)$ be the space of bounded linear operators. Denote by $B_p(H)$ the p-Schatten class

$$B_p(H) = \{v \in B(H) : \|v\|_p^p = \mathrm{tr}((v^*v)^{p/2}) < \infty\},$$

where tr is the usual trace in $B(H)$.

Denote by $U(H)$ the unitary group of H and consider the following classical Banach-Lie group:

$$U_p(H) = \{u \in U(H) : u - 1 \in B_p(H)\},$$

where $1 \in B(H)$ denotes the identity operator. The Lie algebra of $U_p(H)$ can be identified with $B_p(H)_{ah}$, the space of skew-hermitian elements of $B_p(H)$.

Let $S(H) = \{x \in H : \|x\| = 1\}$ be the unit sphere in H. The group $U_p(H)$ acts on $S(H)$,

$$\pi : U_p(H) \times S(H) \to S(H), \qquad u \cdot x := ux.$$

It is clear that this action is smooth and transitive.

The purpose of this paper is to study the Finsler metric induced in $S(H)$ by the action of $U_p(H)$.

For $x \in S(H)$, let $G_x \subset U_p(H)$ be the isotropy group at x, i.e.,

$$G_x := \{u \in U_p(H) : u \cdot x = x\}.$$

The Lie algebra $\mathfrak{g}_x$ of G_x consists of operators w in $B_p(H)$ such that $w^* = -w$ and $w \cdot x = 0$. Consider the quotient p-metric in $T_xS(H)$: if $v \in T_xS(H)$ then

$$\|v\|_{x,p} = \min\{\|z - y\|_p : y \in \mathfrak{g}_x\},$$

where $z \in \mathcal{B}_p(H)_{ah}$ satisfies $(d\pi_x)_1(z) = z \cdot x = v$. We call z a *lifting* of v. A lifting z_0 is called a *minimal lifting* if $\|v\|_{x,p} = \|z_0\|_p$. The quotient norm induces a metric in $S(H)$:

$$d_{S(H),p}(x, ux) = \inf\Big\{L_p(\gamma) := \int_0^1 \|\dot\gamma(t)\|_{\gamma(t),p} : \gamma \subset S(H),\ \gamma(0) = x,\ \gamma(1) = ux\Big\}.$$

We study the problem of finding the geodesic curves, or short paths, for this metric. The results are obtained by applying techniques developed in [2] for abstract homogeneous spaces of $U_p(H)$, to the case of $S(H)$.

This paper is organized as follows. In Section 2 we collect some preliminary facts concerning the geometry of $S(H)$ and $U_p(H)$. The quotient p-metric is introduced in Section 3. In Section 4 we study minimal liftings of $v \in T_xS(H)$. In Section 5 we show the consequence of these facts on the existence and uniqueness of geodesic curves in $S(H)$. Section 6 is devoted to the case $p = 2$, where a reductive homogeneous structure is introduced.

2. Preliminary facts

In this section, we introduce the necessary definitions and we recall certain known facts on the Riemmanian differentiable structure of the sphere $S(H)$, as well as some results on the metric in the unitary group $U_p(H)$.

The tangent space at x, denoted $T_xS(H)$, may be identified with the set of vectors $v \in H$ satisfying $\mathrm{Re}(\langle v, x\rangle) = 0$. By this condition, the double-tangent $TTS(H)$ consists of the set

$$\Big\{(x, v, u, w) \,:\, x \in S(H);\ v, u \in T_xS(H);\ w \in H;\ \mathrm{Re}\big(\langle w, x\rangle + \langle v, u\rangle\big) = 0\Big\}. \quad (1)$$

We consider the metric in $U_p(H)$ given by the length functional L_p:

$$L_p(\alpha) := \int_{t_0}^{t_1} \|\dot\alpha(t)\|_p dt,$$

where $\alpha : [t_0, t_1] \to U_p(H)$ is a piecewise smooth curve. Recall that $\|\cdot\|_p$ denotes the p-norm of operators:

$$\|z\|_p = \mathrm{tr}\left((z^*z)^{p/2}\right)^{1/p}.$$

The rectifiable distance between u_1 and u_2 in U_p is

$$d_p(u_1, u_2) := \inf\{L_p(\alpha) : \alpha \subset U_p(H),\ \alpha \text{ joins } u_0 \text{ and } u_1\}.$$

The following theorem collects several results concerning the rectifiable p-distance in $U_p(H)$. Proofs can be found in [2].

Theorem 2.1. *Let $2 \le p < \infty$. The following facts hold:*

(1) *Let* $u \in U_p(H)$ *and* $v \in B_p(H)_{ah}$ *with* $\|v\| \le \pi$. *Then the curve* $\mu(t) = u\, e^{tv}$, $t \in [0,1]$, *is shorter than any other smooth curve in* $U_p(H)$ *joining the same endpoints. Moreover, if* $\|v\| < \pi$, *this curve is unique with this property.*
(2) *Let* $u_0, u_1 \in U_p(H)$. *Then there exists a minimal geodesic curve joining them. Moreover, if* $\|u_0 - u_1\| < 2$, *this geodesic is unique.*
(3) *There are in* $U_p(H)$ *minimal geodesics of arbitrary length. Thus the diameter of* $U_p(H)$ *is infinite.*
(4) *If* $u, v \in U_p(H)$, *then*

$$\sqrt{1 - \frac{\pi^2}{12}}\, d_p(u,v) \le \|u - v\|_p \le d_p(u,v).$$

In particular, the metric space $(U_p(H), d_p)$ *is complete.*

Next, we recall the following results concerning the geodesic distance. These results are the key in obtaining minimality of geodesics en $S(H)$. Proofs for these statements can be found in [2].

Theorem 2.2. *Let* p *be an even positive integer,* $u \in U_p(H)$ *and* $\beta : [0,1] \to U_p(H)$ *be a non-constant geodesic such that*

$$\beta \subset B_p\left(u, \frac{\pi}{2}\right) = \{w \in U_p(H) : d_p(u,w) < \pi/2\}.$$

Assume further that u *does not belong to any prolongation of* β. *Then*

$$f_p(s) = d_p(u, \beta(s))^p$$

is a strictly convex function.

Corollary 2.3. *Let* $u_1, u_2, u_3 \in U_p(H)$ *with* $u_2, u_3 \in B_p(u_1, \frac{\pi}{4})$ *and assume that they are not aligned (i.e., they do not lie in the same geodesic). Let* γ *be the short geodesic joining* u_2 *with* u_3. *Then* $d_p(u_1, \gamma(s)) < \frac{\pi}{2}$ *for* $s \in [0,1]$ *and* $\frac{\pi}{4}$ *is the radius of convexity of the metric balls of* $U_p(H)$.

3. Quotient p-metric in $S(H)$

In this section, we describe the quotient metric in $S(H)$. Note that $S(H)$ is the orbit of any x in $S(H)$ by the action of $U_p(H)$.

The action of $U_p(H)$ in $S(H)$ induces two kinds of maps. If one fixes $x \in S(H)$, one has the submersion

$$\pi_x : U_p(H) \to S(H), \qquad \pi_x(u) := ux.$$

If one fixes $u \in U_p(H)$, one has the diffeomorphism

$$\ell_u : S(H) \to S(H), \qquad \ell_u(x) := ux.$$

We will consider the orthogonal decomposition induced by x, $H = \langle x \rangle \oplus \langle x \rangle^{\perp}$, in order to describe operators in $B(H)$.

If $x \in S(H)$, the isotropy G_x is the subgroup $\pi^{-1}(x)$, which consists of all operators of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & u_0 \end{pmatrix},$$

where $u_0 : \langle x\rangle^\perp \to \langle x\rangle^\perp$ is an operator in $U_p(\langle x\rangle^\perp)$.

We will denote by $(d\pi_x)_v : B_p(H)_{ah} \to TS(H)$ the differential of π_x at v. In particular, if $v = 1 := \mathrm{Id} \in U_p(H)$ then

$$(d\pi_x)_1(v) = vx.$$

Its kernel is the Lie algebra $\mathfrak{g}_x$. In terms of the decomposition of H with respect to x, an element in $\mathfrak{g}_x$ has the form

$$\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix},$$

where $c : \langle x\rangle^\perp \to \langle x\rangle^\perp$ is skew-hermitian and belongs to $\mathcal{B}_p(\langle x\rangle^\perp)$.

Since $d\pi_x$ is an epimorphism, we can identify the tangent space $T_xS(H)$ with the quotient $\mathcal{B}_p(H)_{ah}/\ker(d\pi_x)$. This viewpoint enables to define the following Finsler metric in $S(H)$:

$$\|v\|_{x,p} = \min\{\|z - y\|_p : y \in \mathfrak{g}_x\},$$

where $z \in \mathcal{B}_p(H)_{ah}$ is any *lifting* of v, i.e. an element such that $(d\pi_x)_1(z) = z{\cdot}x = v$. If z_0 satisfies $\|v\|_{x,p} = \|z_0\|_p$, z_0 is called a minimal lifting. We denote by $\overline{L_p}$ the length functional for piecewise smooth curves in $S(H)$, measured with the quotient norm defined by

$$\overline{L_p}(\gamma) := \int_0^1 \|\dot\gamma\|_{\gamma,p}.$$

As usual the metric distance in $S(H)$ is defined as the infimum of the lengths of the arcs in $S(H)$, namely,

$$\overline{d}_p(x, ux) = \inf\Big\{\overline{L_p}(\gamma) : \gamma \subset S(H),\ \gamma(0) = x,\ \gamma(1) = ux\Big\}.$$

A straightforward computation shows that this metric $\overline{d}_p$ is invariant by the action of $U_p(H)$, i.e., given $u \in U_p(H)$, $x \in S(H)$ and $v \in T_xS(H)$,

$$\|(d\ell_u)_x(v)\|_{ux} = \|v\|_x.$$

In [2] it was proved that if $U_p(H)$ acts transitively and smoothly on a manifold O and we endow the tangent bundle of O with the quotient metric as above, then $(O, \overline{d}_p)$ is complete.

We are interested in describing the minimal liftings of a given $v \in T_xS(H)$. Note that these satisfy

$$\|z_0\|_p \le \|z_0 - y\|_p \quad \text{for all } y \in \mathfrak{g}_x.$$

Let Q be the (non linear) projection $Q : B_p(H)_{ah} \to \overline{\mathfrak{g}_x}^p$ which sends $z \in B_p(H)_{ah}$ to its best approximant $Q(z) \in B_{ah}(H)$ satisfying

$$\|z - Q(z)\|_p \le \|z - y\|_p$$

for all $y \in \bar{\mathfrak{g}_x}^p$. The map Q is continuous and single-valued because $B_p(H)$ is uniformly convex and uniformly smooth (see for instance [5]).

In particular, a minimal lifting z_0 of $v \in T_xS(H)$ belongs to the set

$$\mathfrak{g}_x^{\perp} := Q^{-1}(0) = \{z \in \mathcal{B}_p(H)_{ah} : \|z\|_p \leq \|z-y\|_p \text{ for all } y \in \mathfrak{g}_x\}.$$

4. Characterization of minimal liftings

Note that any $z \in \mathcal{B}_p(H)_{ah}$ can be decomposed as

$$z = z - Q(z) + Q(z),$$

where Q is the (non linear) projection onto $\mathfrak{g}_x$, $z - Q(z) \in \mathfrak{g}_x^{\perp}$ and $Q(z) \in \mathfrak{g}_x$. Since π_x is submersion, the differential $(d\pi_x)_1$ is surjective and then, for any $v \in T_xS(H)$, there exists $z \in \mathcal{B}_p(H)_{ah}$ such that $(d\pi_x)_1(z) = zx = v$. Then a minimal lifting is

$$z_0 = z - Q(z) \in \mathfrak{g}_x^{\perp}.$$

The following theorem establishes the uniqueness of minimal liftings in $S(H)$.

Theorem 4.1. *Let p be a positive even integer, $x \in S(H)$ and $v \in T_xS(H)$. An element $z_0 \in \mathcal{B}_p(H)_{ah}$ such that $z_0x = v$ is a minimal lifting of v if and only if $\operatorname{tr}(z_0^{p-1}y) = 0$ for all $y \in \mathfrak{g}_x$. The lifting z_0 satisfies this condition if and only if its matrix with respect to the decomposition $H = \langle x\rangle \oplus \langle x\rangle$ is*

$$z_0^{p-1} = \begin{pmatrix} \lambda i & -b^* \\ b & 0 \end{pmatrix},$$

where $b : \langle x\rangle \to \langle x\rangle^{\perp}$ and $\lambda \in \mathbb{R}$.

Proof. Suppose that $z_0 \in \mathcal{B}_p(H)_{ah}$ is a minimal lifting. For a fixed $y \in \mathfrak{g}_x$, let $f(t) = \|z_0 - ty\|_p^p$. It is clear that f is a smooth map with a minimum at $t = 0$. Then $f'(0) = 0$. Since $f'(t) = -p \operatorname{tr}((z_0 - ty)^{p-1}y)$, then $\operatorname{tr}(z_0^{p-1}y) = 0$.

Conversely, let $z_0 \in \mathcal{B}_p(H)_{ah}$ be a lifting such that $\operatorname{tr}(z_0^{p-1}y)$ for all $y \in \mathfrak{g}_x$. Suppose that z_0 is not minimal, namely, there is y_0 such that $\|z_0 - y_0\|_p < \|z_0\|_p$. Then, the convex function $f(t) = \|z_0 - ty_0\|_p^p$ (with $f'(0) = 0$) would not have a minimum at $t = 0$, and this is contradiction.

Note that the condition $\operatorname{tr}(z_0^{p-1}y) = 0$ is equivalent to

$$\operatorname{tr}\left(\begin{pmatrix} \lambda i & -b^* \\ b & a \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}\right) = \operatorname{tr}\begin{pmatrix} 0 & 0 \\ 0 & ac \end{pmatrix} = \operatorname{tr}(ac)$$

for all $c \in B_p(\langle x\rangle^{\perp})$ skew-hermitian. Then a is the null operator in $B_p(\langle x\rangle^{\perp})$. $\square$

Corollary 4.2. *Let $x \in S(H)$, $v \in T_xS(H)$ and $p = 2$. Then the unique minimal lifting of v is given by*

$$z_0 = \begin{pmatrix} \lambda i & -v_0^* \\ v_0 & 0 \end{pmatrix},$$

where $v_0 := v - \langle v, x\rangle x \in \langle x\rangle^{\perp}$, $\lambda \in \mathbb{R}$ and $\lambda i = \langle v, x\rangle$.

In the special case when the velocity vector v is (complex) orthogonal to the position x, the minimal lifting z_0 is easy to compute.

Corollary 4.3. *Let $x \in S(H)$, $v \in \langle x\rangle^\perp$ and $p \geq 2$ an even integer. Then $z_0 \in \mathcal{B}_p(H)_{ah}$ is the unique minimal lifting of v in x if and only if it has the form*

$$z_0 = \begin{pmatrix} 0 & -v^* \\ v & 0 \end{pmatrix}$$

with respect to the decomposition $H = \langle x\rangle \oplus \langle x\rangle^\perp$.

Proof. Since $v \in \langle x\rangle^\perp$ and $z_0 \in B_p(H)_{ah}$ satisfies $z_0 x = v$, then

$$\langle z_0 x, x\rangle = \langle v, x\rangle = 0.$$

Then $p_x z_0 p_x = 0$, where p_x denotes the orthogonal projection onto $\langle x\rangle$. Therefore the minimal lifting of v has the matrix form

$$z_0 = \begin{pmatrix} 0 & -a^* \\ a & 0 \end{pmatrix}.$$

It remains to prove that the column a of z_0 is precisely v. Let $\{e_0 = x, e_1, e_2, \dots\}$ be an orthonormal basis of H. The infinite matrix of z_0 in this basis is

$$z_0 = \begin{pmatrix} 0 & -\bar{a_1} & -\bar{a_2} & -\bar{a_3} & \dots \\ a_1 & 0 & 0 & 0 & \dots \\ a_2 & 0 & 0 & 0 & \dots \\ a_3 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since $p \geq 2$ is an even integer, the map $f(s) = s^{p-1}$ induces a bicontinuous map from $\mathcal{B}_p(H)_{ah}$ onto itself. Accordingly, the minimal lifting of v should satisfy

$$z_0^{p-1} = \begin{pmatrix} 0 & -\bar{b_1} & -\bar{b_2} & -\bar{b_3} & \dots \\ b_1 & 0 & 0 & 0 & \dots \\ b_2 & 0 & 0 & 0 & \dots \\ b_3 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $b_i \in \mathbb{C}$.

The relationship between the coefficients of z_0 and z_0^{p-1} is:

$$a_j = \frac{b_j}{(\sum |b_j|^2)^{\frac{p-2}{2}\frac{1}{p-1}}}.$$

Since $z_0 x = v$, this implies:

$$z_0 x = \begin{pmatrix} 0 & -\bar{a_1} & -\bar{a_2} & \bar{a_3} & \dots \\ a_1 & 0 & 0 & 0 & \dots \\ a_2 & 0 & 0 & 0 & \dots \\ a_3 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ v_1 \\ v_2 \\ v_3 \\ \vdots \end{pmatrix} = v,$$

where $v_i = \langle v, e_i\rangle$ for all $i \in \mathbb{N}$. Then, $a_i = v_i$. □

So far we have considered a complex Hilbert space.

Corollary 4.4. *If H is a real Hilbert space and p is a positive even integer, then the unique minimal lifting of $v \in T_x S(H) = \langle x\rangle^\perp$ has the form (with respect to the decomposition $H = \langle x\rangle \oplus \langle x\rangle^\perp$)*

$$z_0 = \begin{pmatrix} 0 & -v^* \\ v & 0 \end{pmatrix}.$$

Let us present some properties of the minimal lifting of v when $\langle v, x\rangle \neq 0$. Since p is even, for $a \in B(H)$, λ is an eigenvalue of a if and only if λ^{p-1} is an eigenvalue of a^{p-1}. Namely, there exists a bijection between the spectrum of a and the spectrum of a^{p-1}.

Lemma 4.5. *Let $x \in S(H)$ and consider the decomposition $H = \langle x\rangle \oplus \langle x\rangle^\perp$. If $m \in \mathcal{B}_p(H)_{ah}$ is*

$$m = \begin{pmatrix} \lambda i & -b^* \\ b & 0 \end{pmatrix},$$

where $0 \neq \lambda \in \mathbb{R}$ and $0 \neq b \in \langle x\rangle^\perp$, then its eigenvalues are $\mu_0 = 0$ and

$$\mu_1 = \frac{\mathrm{sg}(\lambda) i}{2}\left[\sqrt{|\lambda|^2 + 4\|b\|^2} + |\lambda|\right],$$
$$\mu_2 = \frac{\mathrm{sg}(\lambda) i}{2}\left[\sqrt{|\lambda|^2 + 4\|b\|^2} - |\lambda|\right].$$

Moreover, the eigenvectors $\{e_1, e_2\}$ of μ_1, μ_2 are (respectively)

$$e_1 = \frac{\sqrt{2}}{[|\lambda|^2 + 4\|b\|^2]^{\frac{1}{4}}} \begin{pmatrix} \frac{\mathrm{sg}(\lambda) i}{2}[\sqrt{|\lambda|^2 + 4\|b\|^2} + |\lambda|^2]^{\frac{1}{2}} \\ \\ \frac{v}{[\sqrt{|\lambda|^2 + 4\|b\|^2} + |\lambda|^2]^{\frac{1}{2}}} \end{pmatrix},$$
$$e_2 = \frac{\sqrt{2}}{[|\lambda|^2 + 4\|b\|^2]^{\frac{1}{4}}} \begin{pmatrix} \frac{\mathrm{sg}(\lambda) i}{2}[\sqrt{|\lambda|^2 + 4\|b\|^2} - |\lambda|^2]^{\frac{1}{2}} \\ \\ \frac{b}{[\sqrt{|\lambda|^2 + 4\|b\|^2} - |\lambda|^2]^{\frac{1}{2}}} \end{pmatrix}.$$

The nullspace of $\mu = 0$ is $\langle b\rangle^\perp \cap \langle x\rangle^\perp$.

Proof. Note that m is a rank 2 operator, thus $0 \in \sigma(m)$. Let be $w = w_0 + w_1 \in \langle x\rangle \oplus \langle x\rangle^\perp$ such that

$$\begin{pmatrix} \lambda i & -b^* \\ b & 0 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} \lambda i w_0 - b^* w_1 \\ w_0 b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This equality holds if and only if $w_0 = 0$ and $b^* w_1 = 0$. Namely, $\ker(m) = \langle x\rangle^\perp \cap \langle b\rangle^\perp$.

Next, we will prove μ_i, $i = 1, 2$, are eigenvalues of m. Let

$$v^1 := \begin{pmatrix} \mu_1 \\ b \end{pmatrix}.$$

Then

$$mv^1 = \begin{pmatrix} \lambda i & -b^* \\ b & 0 \end{pmatrix} \begin{pmatrix} \mu_1 \\ b \end{pmatrix} = \begin{pmatrix} \lambda i \mu_1 - \|b\|^2 \\ \mu_1 b \end{pmatrix}.$$

$$\mu_1 v^1 = \begin{pmatrix} -\mu_1^2 \\ \mu_1 b \end{pmatrix}$$

It follows that $\mu_1^2 = \lambda i \mu_1 - \|b\|^2$, because

$$\begin{aligned} \mu_1^2 &= \frac{-1}{4}\left[\sqrt{|\lambda|^2 + 4\|b\|^2} + |\lambda|\right]^2 \\ &= \frac{-1}{4}\left[|\lambda|^2 + 4\|b\|^2 + |\lambda|^2 + 2\sqrt{|\lambda|^2 + 4\|b\|^2}|\lambda|\right] \\ &= \frac{-|\lambda|^2}{2} - \|b\|^2 - \frac{|\lambda|}{2}\sqrt{|\lambda|^2 + 4\|b\|^2} \\ &= \frac{-|\lambda|}{2}\left[|\lambda| + \sqrt{|\lambda|^2 + 4\|b\|^2}\right] - \|b\|^2 = \lambda i \mu_1 - \|b\|^2. \end{aligned}$$

These facts imply that $mv^1 = \mu_1 v^1$. The normalization of v^1 is e_1. The other computation is similar. □

Proposition 4.6. *Let $p \geq 2$ be an even integer and $x \in S(H)$, and let $v = \alpha i + a \in TS(H)_x$ (where $a \in \langle x \rangle^\perp$). Then there exists a unitary operator u such that the unique minimal lifting $z_0 \in \mathcal{B}_p(H)_{ah}$ of v in x is*

$$z_0 = U \begin{pmatrix} \sqrt[p-1]{\mu_1} & 0 & 0 & \dots \\ 0 & \sqrt[p-1]{\mu_2} & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} U^{-1}.$$

Proof. Straightforward using 4.1 and 4.5. □

We close this section with the following result, which determines the norm of the minimal liftings.

Proposition 4.7. *Let $x \in S(H)$ and $z_0 \in \mathcal{B}_p(H)_{ah}$, such that*

$$z_0^{p-1} = \begin{pmatrix} \lambda i & -b^* \\ b & 0 \end{pmatrix}$$

with respect to the decomposition $H = \langle x \rangle \oplus \langle x \rangle^\perp$. If $\lambda \neq 0$, then

$$\|z_0\|_p^p = \frac{1}{2^{\frac{p}{p-1}}}\left\{\left[|\sqrt{|\lambda|^2 + 4\|b\|^2} + |\lambda|\right]^{\frac{p}{p-1}} + \left[\sqrt{|\lambda|^2 + 4\|b\|^2} - |\lambda|\right]^{\frac{p}{p-1}}\right\}.$$

If $\lambda = 0$, then

$$\|z_0\|_p^p = 2\|b\|^p.$$

Proof. The proof is apparent. □

Remark 4.8. In the case $p=2$, the above result shows the difference between the quotient metric and the usual metric in $S(H)$. If γ is a curve parametrized in the interval $[0,1]$ with constant velocity, then

$$\left[\overline{L}_2(\gamma)\right]^2 = \left[\int_0^1 \|v\|_x\right]^2 = \|z_0\|_2^2 = |\lambda|^2 + 2\|b\|^2$$

in the quotient metric. However, its length measured with usual metric is

$$[L_2(\gamma)]^2 = \left[\int_0^1 \|v\|\right]^2 = \|v\|^2 = |\lambda|^2 + \|b\|_2^2.$$

This shows that the quotient metric in the sphere of a complex Hilbert space is different from the usual metric. If H is a real Hilbert space, both metrics coincide.

5. Minimality of geodesic curves in $S(H)$

Let $ad_a : B_p(H) \to B_p(H)$ be the operator $ad_a(x) := xa - ax$.

Proposition 5.1. *Let $x \in S(H)$ and $Q = Q_{\mathfrak{g}_x}$ the best approximant projection. Let $\gamma(t) := \Gamma(t)x \subset S(H)$, where $\Gamma : [0,1] \to U_p(H)$ is a piecewise C^1 curve. Put $F(z) := \dfrac{e^z - 1}{z}$. Then there exists a piecewise C^1 curve $z : [0,1] \to \mathfrak{g}_x$ with $z(0) = 0$ such that*

$$F(ad_z)\dot{z} = -Q(\Gamma^*\dot{\Gamma}).$$

If $u_\Gamma = e^z \in G_x$ then $u_\Gamma(t) \in B_p(H)$ is a solution of the differential equation

$$\dot{u}_\Gamma u_\Gamma^* = -Q(\Gamma^*\dot{\Gamma})$$

and $L_p(u_\Gamma) \leq 2L_p(\Gamma)$.

Proof. See [2]. □

Remark 5.2. Let $x \in S(H)$ and $\gamma := \Gamma x \subset S(H)$ parametrized in the interval $[0,1]$. Let u_γ be the curve of the previous proposition. Since $u_\Gamma \in G_x$, we have $\Gamma u_\Gamma x = \gamma$. Moreover, by this same proposition $L_p[\beta] = \overline{L}[\gamma] \leq L_p(\Gamma)$. The curve β is called an isometric lifting of γ.

Theorem 5.3. *Let p be an even positive integer, $x \in S(H)$, $v \in T_xS(H)$ and $z_0 \in \mathcal{B}_p(H)_{ah}$ the unique minimal lifting of v. Let $\mu : [0,1] \to S(H)$ be the curve*

$$\mu(t) = e^{tz_0}x$$

which satisfies $\mu(0) = x$ and $\dot{\mu}(0) = v = \alpha i + v_0 \in H = \langle x\rangle \oplus \langle x\rangle^\perp$. If

$$\|z_0\|_p \leq \frac{\pi}{4}$$

then the curve μ is shorter than any other curve in $S(H)$ joining the same endpoints.

Proof. The proof is based on the existence of minimal lifting of curves (in Remark 5.2) and the convexity of the maps $f_p(t) = d_p(1, e^{z_0}e^{ty})$ for any $y \in \mathfrak{g}_x$ (by Theorem 2.2). This theorem was proved in [2] for homogeneous manifolds on which $U_p(H)$ acts transitively and smoothly and the group G_x is locally exponential. □

The previous theorem establishes conditions which guarantee that a short arc of the curve $\gamma(t) := e^{tz}x$ minimizes length among all curves with the same endpoints.

When $v \in \langle x\rangle^\perp$, by Corollary 4.3, the minimal lifting of v has matrix form, with respect to $H = \langle x\rangle \oplus \langle x\rangle^\perp$,

$$z_0 = \begin{pmatrix} 0 & -v^* \\ v & 0 \end{pmatrix}.$$

Note that this lifting is independent of $p \geq 2$. Hence, we obtain a uniform bound for all p even in terms of $\|v\|$, in order that the curve $\gamma(t) := e^{tz_0}x$ is short.

Theorem 5.4. *Let p be an even positive integer, $x \in S(H)$, $v \in \langle x\rangle^\perp \subset T_xS(H)$ and $z_0 \in \mathcal{B}_p(H)_{ah}$ be the unique minimal lifting of v. If*

$$\|v\| \leq \frac{\pi}{4\sqrt[p]{2}}$$

then the curve $\mu(t) = e^{tz_0}x$, which satisfies $\mu(0) = x$ and $\dot\mu(0) = v$, is minimal in the interval $[0,1]$. Moreover, if $\|v\| < \frac{\pi}{4\sqrt{2}}$, this curve is short for all p-quotient metrics (p an even integer).

Proof. The result follows from Corollary 5.3 and Theorem 4.7. □

Corollary 5.5. *Let H be a real Hilbert space and $p \geq 2$ an even integer. Given $x \in S(H)$, the minimal lifting of $v \in T_xS(H)$ defines a minimal geodesic in the interval $[0,1]$ if*

$$\|v\| \leq \frac{\pi}{4\sqrt[p]{2}}.$$

6. The case $p = 2$

In this section, we describe the quotient 2-metric of the sphere $S(H)$. $S(H)$ is an infinite dimensional homogeneous reductive space. The geometry of these spaces has been studied in [6] in the C^*-algebra context. From this reference we take definitions and calculations.

Given $x \in S(H)$, we consider the decomposition $H = \langle x\rangle \oplus \langle x\rangle^\perp$ and the matrix form of the operators in terms of this decomposition.

We define the metric induced by the decomposition

$$\mathcal{B}_2(H)_{ah} = \mathfrak{g}_x \oplus \mathfrak{g}_x^\perp,$$

where $\mathfrak{g}_x$ is the Lie algebra of G at $x \in S(H)$.

Consider again the map π_x and its derivative $(d\pi_x)_1$. We denote by

$$\delta_x := (d\pi_x)_1 \,|_{\mathfrak{g}_x^\perp} \colon \mathfrak{g}_x^\perp \to T_xS(H)$$

given by $\delta_x(z) := zx$. This map is a linear bounded isomorphism between these spaces. Then, we can define its inverse

$$\kappa_x(v) := z, \quad \text{if } (\delta_x)_1(z) = v.$$

By Corollary 4.2, $\kappa_x(v)$ has matrix form

$$\kappa_x(v) := \begin{pmatrix} \lambda i & -v_0^* \\ v_0 & 0 \end{pmatrix},$$

where $v = i\lambda x + v_0$, $v_0 \in \langle x\rangle^\perp$, and $\lambda = \operatorname{Im}\langle v, x\rangle \in \mathbb{R}$.

Let $z, w \in H$. Denote $z \otimes w \in B(H)$ the elementary rank one operator

$$z \otimes w(h) := w^* z(h) = \langle h, w\rangle z.$$

We can write κ_x as

$$\kappa_x(v) = v \otimes x - x \otimes v - \langle v, x\rangle x \otimes x.$$

By definition, it is clear that $\delta_x \circ \kappa_x = \operatorname{Id}_{T_x S(H)}$ and $\kappa_x \circ \delta_x = P_{\mathfrak{g}^\perp}$ the orthogonal projection onto $\mathfrak{g}^\perp$. Indeed,

$$\begin{aligned}\delta_x \circ \kappa_x(v) &= \delta_x(v \otimes x - x \otimes v - \langle v, x\rangle x \otimes x)\\ &= -\langle x, v\rangle x + \|x\|^2 v - \langle v, x\rangle x = v.\end{aligned}$$

Here we use that $\langle v, x\rangle = -\langle x, v\rangle$, because $v \in T_x S(H)$, i.e. $\operatorname{Re}\langle v, x\rangle = 0$.

To prove the other equality, note that the projection onto $\mathfrak{g}_x^\perp$ is given by

$$P_{\mathfrak{g}^\perp}(z) = p_x z p_x + (1 - p_x) z p_x + p_x z(1 - p_x),$$

where p_x is the projection onto $\langle x\rangle$ given by $x \otimes x$. Then

$$\begin{aligned}P_{\mathfrak{g}^\perp}(z) &= (x \otimes x)z(x \otimes x) + (1 - x \otimes x)z(x \otimes x) + (x \otimes x)z(1 - x \otimes x)\\ &= z - (1 - x \otimes x)z(1 - x \otimes x).\end{aligned}$$

Then,

$$\begin{aligned}\kappa_x \circ \delta_x(z) &= \kappa_x(zx)\\ &= z(x \otimes x) - (x \otimes x)z^* - \langle zx, x\rangle(x \otimes x)\\ &= (x \otimes x)z(x \otimes x) + (1 - (x \otimes x))z(x \otimes x) + (x \otimes x)z - (x \otimes x)(zx \otimes x)\\ &= p_x z p_x + (1 - p_x) z p_x + p_x z(1 - p_x) = P_{\mathfrak{g}^\perp} z.\end{aligned}$$

Note that the decomposition $\mathcal{B}_2(H)_{ah} = \mathfrak{g}_x \oplus \mathfrak{g}_x^\perp$ is equivariant under conjugation with u in $G_x \subset U_2(H)$. Namely,

$$uvu^{-1} \in \mathfrak{g}_x^\perp \qquad \text{if } v \in \mathfrak{g}_x^\perp \text{ and } u \in G_x.$$

Indeed, using the matrix representation with respect to $H = \langle x\rangle \oplus \langle x\rangle^\perp$, let $u_0 \in U_p(\langle x\rangle^\perp)$ and $v_0 \in H$ such that

$$u = \begin{pmatrix} 1 & 0 \\ 0 & u_0 \end{pmatrix}, \qquad v = \begin{pmatrix} \lambda i & -v_0^* \\ v_0 & 0 \end{pmatrix};$$

then

$$\begin{aligned}uvu^{-1} &= \begin{pmatrix} 1 & 0 \\ 0 & u_0 \end{pmatrix}\begin{pmatrix} \lambda i & -v_0^* \\ v_0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & u_0^{-1} \end{pmatrix}\\ &= \begin{pmatrix} \lambda i & -v_0^* \\ u_0 v_0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & u_0^{-1} \end{pmatrix}\\ &= \begin{pmatrix} \lambda i & -v_0^* u_0^{-1} \\ u_0 v_0 & 0 \end{pmatrix},\end{aligned}$$

and this operator lies in $\mathfrak{g}_x^\perp$.

Summarizing, we constructed a map $\kappa_x : T_x S(H) \to \mathfrak{g}_x^\perp$ such that

- $(d\pi_x)_1 \circ \kappa_x : T_xS(H) \to T_xS(H)$ is the identity mapping,
- $\kappa_x(T_xS(H))$ is ad_u-invariant for $u \in G_x$.

This mapping allows us to induce a metric in $S(H)$. Given $x \in S(H)$, we define the structure 1-form by

$$\mathcal{K} : T_yS(H) \to \mathcal{B}_2(H)_{ah}, \qquad \mathcal{K}(y) := ad_u \circ \kappa_x \circ (d\ell_u)^{-1} \quad \text{if } \ell_u x = ux = y.$$

The inner product in $T_xS(H)$ is given by

$$\langle v, w\rangle_x = \text{Re tr}\,(\kappa_x(w)^*\kappa_x(v)) = -\,\text{tr}\,(\kappa_x(w)\kappa_x(v))\,,$$

where Re tr denotes the real part of the trace of operators in $B(H)$.

In terms of elementary rank one operators

$$\begin{aligned}\langle v, w\rangle_x &= -\,\text{Re tr}(\kappa_x(w)\kappa_x(v))\\ &= \langle w, x\rangle\langle v, x\rangle \,\text{tr}(x \otimes x) + \text{Re}\langle v, w\rangle \,\text{tr}(x \otimes x) + \text{tr}(w \otimes v)\\ &= \langle w, x\rangle\langle v, x\rangle + 2\,\text{Re}\langle v, w\rangle.\end{aligned}$$

Note that if z is a lifting of $v \in T_xS(H)$ then

$$\langle v, v\rangle_x = \text{tr}(\kappa_x(v)^2) = \|\kappa_x(v)\|_2^2 = \|\kappa_x(\delta_x(z))\|_2^2 = \|z\|_2^2 = \|v\|_x^2.$$

Therefore, the metric induced by the inner product is the quotient 2-metric defined in previous sections.

We can define a horizontal lifting of a curve on $S(H)$ to $U_2(H)$ as follows: given $\gamma(t) \subset S(H)$ ($t \in I$, an interval with $0 \in I$; $\gamma(0) = x$), there is $\Gamma \subset U_2(H)$ ($t \in I$) such that $\Gamma(0) = 1$ and it satisfies

$$\dot{\Gamma} = \kappa_{\gamma(t)}(\dot{\gamma}(t))\Gamma(t). \tag{2}$$

This equation is called parallel transport equation for γ. A solution Γ satisfies

$$\begin{aligned}&\Gamma(t) \in U_2(H), \quad t \in [0, 1]\\ &\pi_\gamma(\Gamma) = \gamma \quad (\Gamma \text{ lifts } \gamma)\\ &\Gamma^*\Gamma \in \mathfrak{g}_\gamma \quad (\Gamma \text{ is horizontal}).\end{aligned}$$

Let $x \in S(H)$ and consider the curve $\gamma : [0, 1] \to S(H)$,

$$\gamma(t) := \cos(kt)x + \frac{\sin(kt)}{k}v,$$

where $v \in T_xS(H)$.

Note that if $k = \|v\|$, γ describes an arc of a maximal circle that satisfies $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. Its length with respect to the usual metric is k. If we choose to take $v = y - x - \text{Re}\langle y - x, x\rangle x$ for any $y \in S(H)$, $y \neq -x$ and $k = \arccos(\text{Re}\langle y, x\rangle) \in [-\pi, \pi]$, then γ is the arc of a maximal circle joining x with y and its length is $|k|$. Note that these curves are precisely the great circles (intersections of $S(H)$ with 2-planes through the origin) with constant velocity parametrizations.

In this case,

$$\begin{aligned}\kappa_{\gamma(t)}(\dot\gamma(t)) &= \dot\gamma(t)\otimes\gamma(t) - \gamma(t)\otimes\dot\gamma(t) - \langle\dot\gamma(t),\gamma(t)\rangle\gamma(t)\otimes\gamma(t)\\ &= x\otimes v - v\otimes x\\ &\quad + \langle x,v\rangle\Big\{\cos^2(kt)(x\otimes x) + \frac{\sin^2(kt)}{k^2}(v\otimes v) + \frac{\sin(2kt)}{2k}[x\otimes v + v\otimes x]\Big\}.\end{aligned} \tag{3}$$

Then, we deduce the following lemma.

Lemma 6.1. *Let $x \in S(H)$, $v \in \langle x\rangle^\perp \subset T_xS(H)$ and $k = \|v\|$. Let $\gamma : [0,1] \to S(H)$ be the curve that satisfies $\gamma(0) = x, \dot\gamma(0) = v$ and is given by*

$$\gamma(t) := \cos(kt)x + \frac{\sin(kt)}{k}v.$$

Then the parallel transport of the curve γ is the solution of

$$\begin{cases}\dot\Gamma(t) = \kappa_x(v)\Gamma\\ \Gamma(0) = 1.\end{cases}$$

Proof. Using $\langle v,x\rangle = 0$, the proof follows from (3) in (2). □

Lemma 6.2. *Let $x, y \in S(H)$ such that $\langle y,x\rangle \in \mathbb{R}$. Let $v = y - x - \mathrm{Re}\langle y-x,x\rangle x$ and $k = \langle y,x\rangle$. Let $\gamma : [0,1] \to S(H)$ be a curve that satisfies $\gamma(0) = x$, $\gamma(1) = y$,*

$$\gamma(t) := \cos(kt)x + \frac{\sin(kt)}{k}v.$$

Then, the parallel transport of this curve is the solution of

$$\begin{cases}\dot\Gamma(t) = \kappa_x(v)\Gamma\\ \Gamma(0) = 1.\end{cases}$$

Proof. As in the above lemma, it suffices to see that if $\langle y,x\rangle \in \mathbb{R}$ then $v \in \langle x\rangle^\perp$:

$$\langle y - x - \mathrm{Re}\langle y-x,x\rangle x, x\rangle = \langle y - \langle y,x\rangle x, x\rangle = 0. \qquad \square$$

Next we analyse the natural connection which is induced by the quotient metric. In [6], two natural connections were introduced.

The first connection is called the reductive connection ∇^r. For each $x \in S(H)$, this connection is given by

$$\kappa_x(\nabla^r_w V(x)) := \kappa_x(w)(\kappa_x V(x)) + [\kappa_x(V(x)), \kappa_x(w)],$$

where V is a tangent field and $w \in T_xS(H)$. Here $Y(X)$ denotes the derivative of X in the direction of Y and $[X,Y]$ is the commutator of operators in $B(H)$.

Proposition 6.3. *The reductive connection ∇^r is compatible with the quotient metric in $S(H)$. Given $x \in S(H)$, V a tangent field and $w \in T_xS(H)$, the connection is given by*

$$\nabla^r_w(V) = \dot V w - \langle V,x\rangle w + \big[\langle v,x\rangle\langle w,x\rangle - \langle w,v\rangle\big]x. \tag{4}$$

Proof. Let $x \in S(H)$ and $w \in T_xS(H)$ and consider $\beta : I \to TS(H)$ the curve given by $\beta(t) = (x_t, w_t)$ such that $x_0 = x$ and $\dot{x}_0 = w$. Note that

$$\begin{aligned}
\kappa_x(w)(\kappa_x(V(x))) = & \left.\frac{d}{dt}\kappa_{x_t}(V(x_t))\right|_{t=0} \\
= & \left.\frac{d}{dt}(V(x_t)) \otimes x_t + V(x_t) \otimes \dot{x}_t - \dot{x}_t \otimes V(x_t) - x_t \otimes \frac{d}{dt}(V(x_t))\right|_{t=0} \\
- & \left.\left[\langle \frac{d}{dt}(V(x_t)), x_t\rangle + \langle V(x_t), \dot{x}_t\rangle\right] x_t \otimes x_t\right|_{t=0} \\
- & \langle V(x_t), x_t\rangle (x_t \otimes \dot{x}_t + \dot{x}_t \otimes x_t)|_{t=0}
\end{aligned}$$

Using the notation $V := V(x_0)$ and $\frac{d}{dt}(V(x_t))|_{t=0} = DV(x_0)\dot{x}_0 =: \dot{V}w$ we obtain

$$\begin{aligned}
\kappa_x(w)(\kappa_x(V(x))) = \dot{V}w \otimes x + V \otimes w - w \otimes V - x \otimes \dot{V}w \\
- [\langle \dot{V}w, x\rangle + \langle V, w\rangle] x \otimes x - \langle V, x\rangle (x \otimes w + w \otimes x). \quad (5)
\end{aligned}$$

The latter is due to the fact that the curve $\beta(t) \in TS(H)$, and then $\dot{\beta}(t) = (x_t, V(x_t); \dot{x}_t, \dot{V}(x_t)) \in TTS(H)$, i.e., its derivative satisfies (1):

$$\mathrm{Re}\big(\langle \dot{V}w, x\rangle + \langle v, w\rangle\big) \;=\; 0.$$

On the other hand, the commutator between $\kappa_x(V)$ and $\kappa_x(w)$ is

$$\begin{aligned}
[\kappa_x(V), \kappa_x(w)] &= \kappa_x(V)\kappa_x(w) - \kappa_x(w), \kappa_x(V) \\
&= w \otimes V - V \otimes w + [\langle V, w\rangle - \langle w, V\rangle]\, x \otimes x.
\end{aligned} \quad (6)$$

Reordering (5) and (6), we obtain the formula

$$\begin{aligned}
\kappa_x(\nabla^r_w(V))) = \dot{V}w \otimes x - x \otimes \dot{V}w - \langle V, x\rangle \big[w \otimes x + w \otimes x\big] \\
- \Big[\langle w, V\rangle + \langle \dot{V}w, x\rangle\Big]\, x \otimes x.
\end{aligned}$$

Note that $z = k_x(\nabla^r_w(V_x))$ satisfies $z \in \mathfrak{g}_x^{\perp}$, namely, $z = -z^*$ and $(1 - x \otimes x)z(1 - x \otimes x) = 0$. Then

$$\begin{aligned}
\nabla^r_w(V) &= \delta_x(\kappa_x(\nabla^r_w(V))) \\
&= \dot{V}w - \langle x, \dot{V}w\rangle x - \langle V, x\rangle \big[w + \langle x, w\rangle x\big] - \big[\langle w, v\rangle + \langle \dot{V}w, x\rangle\big] x \\
&= \dot{V}w - \langle V, x\rangle w + \big[\langle v, x\rangle\langle w, x\rangle - \langle w, v\rangle\big] x.
\end{aligned}$$

Since the mappings κ_x are isometries, the reductive connection is compatible with the quotient metric. □

The second connection is called the classifying connection ∇^c. Using the above notations, this connection is defined by

$$\begin{aligned}
\kappa_x(\nabla^c_w(V)) &= P^x_{\mathfrak{g}^{\perp}}(\kappa_x(w)(\kappa_x(V(x)))) \\
&= \kappa_x(w)(\kappa_x(V_x)) - (1 - x \otimes x)\kappa_x(w)(\kappa_x(V_x))(1 - x \otimes x),
\end{aligned} \quad (7)$$

where V is a tangent field over $S(H)$ and $w \in T_xS(H)$.

Proposition 6.4. *The classifying connection ∇^c is compatible with the quotient metric in $S(H)$. For $x \in S(H)$, V a tangent field and $w \in T_xS(H)$, this connection is given by*

$$\nabla^c_w(V) = \dot{V}w + \big[\langle V, w\rangle - \langle w, x\rangle\langle V, x\rangle\big]x - \langle w, x\rangle V. \tag{8}$$

Proof. Using calculations similar to those in the above proposition (formula (7)), we can write

$$\begin{aligned}\kappa_x(\nabla^c_w(V)) &= \dot{V}w \otimes x - x \otimes \dot{V}w \\ &\qquad - \langle w, x\rangle[V \otimes x + x \otimes V] - [\langle \dot{V}w, x\rangle + \langle V, w\rangle]x \otimes x.\end{aligned}$$

Then

$$\begin{aligned}\nabla^c_w(V) &= \delta_x(\kappa_x(\nabla^c_w(V))) \\ &= \dot{V}w - \langle x, \dot{V}w\rangle x - \langle w, x\rangle[V + \langle x, V\rangle x] - [\langle \dot{V}w, x\rangle + \langle V, w\rangle]x \\ &= \dot{V}w + \big[\langle V, w\rangle - \langle w, x\rangle\langle V, x\rangle\big]x - \langle w, x\rangle V.\end{aligned}$$

The compatibility of this connection with the quotient metric was proved in [2]. □

Remark 6.5. The classifying connection (8) has the same geodesics as the reductive connection (4). These connections have opposite torsion (see [6]). Then we can define $\nabla = \frac{1}{2}(\nabla^r + \nabla^c)$. This new connection is symmetric and it has the same geodesics as ∇^r and ∇^r.

The following result summarizes these remarks.

Proposition 6.6. *Using the same hypothesis and notations as in the above propositions, the Levi-Civita connection for the quotient metric is given by*

$$\nabla_w(V) = \dot{V}w - \frac{1}{2}\big[\langle v, x\rangle w + \langle w, x\rangle v\big], \tag{9}$$

for $x \in S(H)$ and $v \in T_xS(H)$. The geodesic curve starting at x with velocity v is given by

$$\gamma(t) = e^{\kappa_x(v)t}x.$$

Proof. The connection $\nabla = \frac{1}{2}(\nabla^r + \nabla^c)$ is compatible with the quotient metric because the reductive connection and the classifying connection are compatible. □

Using Propositions 6.1 and 6.2, we can prove the following results.

Proposition 6.7. *Let $x \in S(H)$.*

- *Let $v \in T_xS(H)$ such that $v \in \langle x\rangle^\perp$. Then the curve $\gamma : [0,1] \to S(H)$ given by*

$$\gamma(t) := \cos(\|v\|t)x + \frac{\sin(\|v\|t)}{\|v\|}v$$

is the geodesic curve of the homogeneous structure in $S(H)$, which satisfies $\gamma(0) = x$, $\dot{\gamma}(0) = v$ and $L(\gamma) = |k|$.

- *Let $y \in S(H)$, $y \neq -x$ such that $\langle y, x\rangle \in \mathbb{R}$. Define $v = y - \langle x, y\rangle x$ and $k = \arccos(\langle y, x\rangle)$. Then the curve $\gamma : [0, 1] \to S(H)$ given by*
$$\gamma(t) := \cos(kt)x + \frac{\sin(kt)}{\|v\|}v$$
is the geodesic curve of the homogeneous structure in $S(H)$, which joins x to y with length $L(\gamma) = k$.
- *Let $y \in S(H)$, $y = e^{\theta i}x$ such that $\theta \in [0, \frac{\pi}{4}]$. Then the curve $\gamma : [0, 1] \to S(H)$ given by*
$$\gamma(t) := e^{\theta it}x$$
is the geodesic curve of the homogeneous structure in $S(H)$, which joins x to y with length $L(\gamma) = \theta$.

Corollary 6.8. *If H is a real Hilbert space, the geodesic curves of the reductive structure are precisely the great circles with constant velocity parametrization.*

References

[1] Andruchow, E.; Chiumento, E.; Larotonda, G.: *Homogeneous manifolds from noncommutative measure spaces*, J. Math. Anal. Appl. 365 (2010), 541–558. MR 2587057.
[2] Andruchow, E; Larotonda, G.; Recht, L.: *Finsler geometry and actions of the p-Schatten unitary groups*, Trans. Amer. Math. Soc. 362 (2010), 319–344. MR 2550153.
[3] Lang, S.: *Differential and Riemannian manifolds.* Third edition. Graduate Texts in Mathematics, 160. Springer-Verlag, New York, 1995. MR 1335233.
[4] Larotonda, G.: *Estructuras geométricas para las variedades de Banach* [PDF document]. Retrieved from `http://glaroton.ungs.edu.ar/estructuras.pdf`.
[5] Li, C.; Wang, X.; Yang, W.: *An estimate for Lipschitz constants of metric projections*, J. Math. Anal. Appl. 231 (1999), 133–141. MR 1676721.
[6] Mata-Lorenzo, L. E.; Recht, L.: *Infinite-dimensional homogeneous reductive spaces.* Acta Cient. Venezolana 43 (1992), 76–90. MR 1185114.

E. Andruchow
Instituto de Ciencias, Universidad Nacional de Gral. Sarmiento
J. M. Gutiérrez 1150, 1613 Los Polvorines, Argentina, and
Instituto Argentino de Matemática
Saavedra 15, 3er. piso, 1083 Buenos Aires, Argentina
eandruch@ungs.edu.ar

A. C. Antunez ✉
Instituto de Ciencias, Universidad Nacional de Gral. Sarmiento
J. M. Gutiérrez 1150, 1613 Los Polvorines, Argentina
aantunez@ungs.edu.ar

Received: August 11, 2015
Accepted: December 18, 2015