Linear Algebra and its Applications 614 (2021) 437–454

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

The determinant of the distance matrix of graphs with blocks at most bicyclic

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ARTICLE INFO

Article history:
Received 11 December 2019
Accepted 11 November 2020
Available online 18 November 2020
Submitted by V. Trevisan

MSC:
15A15
15A24

Keywords:
Bicyclic graphs
Determinant
Distance matrix

ABSTRACT

Let G be a connected graph on n vertices and $D(G)$ be its distance matrix. The formula for computing the determinant of this matrix in terms of the number of vertices is known when the graph is either a tree or a unicyclic graph. In this work we generalize these results, obtaining the determinant of the distance matrix of any graph whose block decomposition consists of edges, unicyclic and bicyclic graphs.

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[1] Work partially supported by ANPCyT PICT 2017-1315.
[2] Work partially supported by UNS Grants PGI 24/L103 and PGI 24/L115.
[3] Work partially supported by MATH-AmSud 18-MATH-01.
[4] Work partially supported by CNPq Grants 408494/2016-6 and 305411/2016-0.

https://doi.org/10.1016/j.laa.2020.11.008
0024-3795/

1. Introduction

A *graph* $G = (V, E)$, or simply G, consists of a nonempty set V of *vertices* and a set E of *edges*, formed by 2-element subsets of V. We will consider graphs without multiple edges and without loops. Let G be a connected graph on n vertices with vertex set $V = \{v_1, \ldots, v_n\}$. The *distance between vertices* v_i and v_j, denoted $d(v_i, v_j)$, is the number of edges of a shortest path from v_i to v_j. The *distance matrix* of G, denoted $D(G)$, is the $n \times n$ symmetric matrix having its (i, j)-entry equal to $d(v_i, v_j)$. We also use $d_{i,j}$ to denote $d(v_i, v_j)$.

The distance matrix has been widely studied in the literature. The interest in this matrix was motivated by the connection with a communication problem (see [3,5] for more details). In an early article, Graham and Pollack [3] presented a remarkable result, proving that the determinant of the distance matrix of a tree T on n vertices only depends on n, being equal to $(-1)^{n-1}(n-1)2^{n-2}$. This result was generalized by Graham, Hoffman, and Hosoya in 1977 [4], who proved that, for any graph G, the determinant of $D(G)$ depends only on the blocks of G (see Theorem 2).

In 2005, more than 30 years after the result of Graham and Pollack on trees, Bapat, Kirkland and Neumann [1] exhibited a formula for the determinant of the distance matrix of a unicyclic graph (see Theorem 1).

For a bicyclic graph, the determinant can be easily computed in the case where the cycles have no common edges, since its blocks are edges and cycles. In a conference article [2], we presented some advances for the remaining cases; i.e., when the cycles share at least one edge. In addition, we conjectured the formula for the remaining cases. In the present article, we completely solve these conjectures, extending the formula of the determinant of $D(G)$ to graphs G having bicyclic blocks as well as trees and unicyclic blocks.

This paper is organized as follows. In Section 2 we present some basic notations, preliminary results, and we briefly describe previous results in connection with the determinant of the distance matrix of a bicyclic graph. In Section 3 we examine the determinant of the distance matrix for special classes of bicyclic graphs, namely θ-graphs, whose definition is stated in Section 2, and also the determinant of a θ-graph attached to a path. In Theorem 5, we present a formula for the determinant of a graph where each block is generated from a tree by the addition of at most two edges (graphs with blocks at most bicyclic).

2. Definitions and preliminary results

A *tree* is a connected acyclic graph. A *unicyclic graph* is a connected graph with as many edges as vertices. The path and the cycle on n vertices are denoted by P_n and C_n, respectively.

The determinant and the sum of all cofactors of the distance matrix of a cycle are already known. Recall that, for any square matrix A, the cofactor $c_{i,j}$ is defined as

$(-1)^{i+j}$ times the determinant of the submatrix obtained by deleting its ith row and jth column. Let $\operatorname{cof}(A) = \sum_{i,j} c_{i,j}$ be the sum of the cofactors of A.

Lemma 1 *([1,7]). For each $n \geq 3$:*

- *if n is odd,* $\det D(C_n) = (n^2 - 1)/4$ *and* $\operatorname{cof} D(C_n) = n$*;*
- *if n is even,* $\det D(C_n) = 0$ *and* $\operatorname{cof} D(C_n) = 0$*.*

In [1], the determinant of $D(G)$ was obtained when G is a unicyclic graph.

Theorem 1 *([1]). Let G be a unicyclic graph consisting of a cycle of length l plus m edges outside the cycle. If l is even, then* $\det D(G) = 0$*; otherwise:*

$$\det D(G) = (-2)^m \frac{l^2 + 2ml - 1}{4}.$$

A *cut-vertex* of a connected graph is a vertex whose removal disconnects the graph. A *block of a graph* G is a maximal connected subgraph of G having no cut-vertices. A *block* is a connected graph having no cut-vertices.

In [4], it was proved that, if the blocks of a graph G are $G_1, G_2, \ldots, G_k$, then $\det D(G)$ depends only on $\det D(G_1), \det D(G_2), \ldots, \det D(G_k)$ and $\operatorname{cof} D(G_1)$, $\operatorname{cof} D(G_2), \ldots,$ $\operatorname{cof} D(G_k)$.

Theorem 2 *([4]). If G is a connected graph whose blocks are $G_1, G_2, \ldots, G_k$, then*

$$\det D(G) = \sum_{i=1}^{k} \det D(G_i) \prod_{j \in \{1,2,\ldots,k\}-\{i\}} \operatorname{cof} D(G_j)$$

and

$$\operatorname{cof} D(G) = \prod_{i=1}^{k} \operatorname{cof} D(G_i).$$

A *cactus* is a connected graph in which any two cycles have at most one vertex in common. By definition, every unicyclic graph is a cactus. Moreover, each block of a cactus on at least two vertices is either an edge or a cycle. As $\det D(G)$ depends only on the blocks of G and $\det D$ and $\operatorname{cof} D$ are known for an edge and for the cycles, we obtain the next corollary as an immediate consequence of Lemma 1 and Theorem 2.

Corollary 1. *Let G be a connected cactus having precisely c cycles whose lengths are $l_1, l_2, \ldots, l_c$ plus m other edges outside these cycles.*

- *If at least one of $l_1, l_2, \ldots, l_c$ is even, then* $\det D(G) = 0$*.*

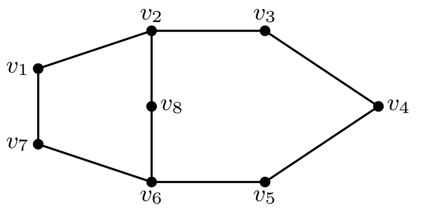

Fig. 1. $\theta(2, 3, 4)$.

- *Otherwise (i.e., if all of $l_1, l_2, \ldots, l_c$ are odd),*

$$\det D(G) = (-2)^m \left(\prod_{i=1}^{c} l_i\right) \left(\frac{m}{2} + \sum_{i=1}^{c} \frac{l_i^2 - 1}{4l_i}\right).$$

A *bicyclic graph* is a graph obtained by adding an edge to a unicyclic graph. The special case of $c = 2$ in the formula of the corollary above was also obtained in [6] by alternative means, corresponding to a special class of bicyclic graphs.

As $\det D$ for all cacti is known, in order to find $\det D$ for all bicyclic graphs, it is enough to find $\det D$ and $\operatorname{cof} D$ for bicyclic blocks.

Definition 1. Let $P_{l+1}, P_{p+1}, P_{q+1}$ be three vertex disjoint paths, $l \geq 1$ and $p, q \geq 2$, each of them having endpoints, $v_1^l, v_2^l, v_1^p, v_2^p, v_1^q, v_2^q$, respectively. We denote by $\theta(l, p, q)$-graph, or simply θ-graph, the graph obtained by identifying the vertices v_1^l, v_1^p, v_1^q as one vertex, and proceeding in the same way for v_2^l, v_2^p, v_2^q (see Fig. 1).

Note that $\theta(l, p, q)$-graph is a bicyclic graph, with no pendant vertex, whose cycles share at least one edge. In [2], we proved the following results:

Proposition 1 *([2, Lemma 3.1]). For every positive integer k,*

$$\det D(\theta(2, 2, 2k + 1)) = 4(k^2 + k - 1).$$

Proposition 2 *([2, Lemma 3.2]). Let G be one of the graphs below:*

- *$\theta(1, 2k - 1, 2k - 1)$, for $k \geq 2$;*
- *$\theta(2, 2, 2k - 2)$, for $k \geq 3$;*
- *$\theta(l, p, q)$, for $l \geq 2$, $p \geq 3$, and $q \geq 3$.*

Then $\det D(G) = 0$.

3. Bicyclic graphs

The next theorem yields the determinant of $D(G)$ when $G = \theta(l, p, q)$, completing the remaining cases in [2].

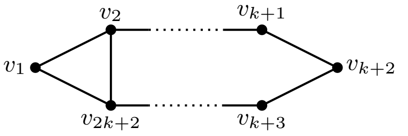

Fig. 2. $\theta(1, 2, 2k)$.

Theorem 3. *Let $G = \theta(l, p, q)$. The following assertions hold:*

(a) *If $G = \theta(1, p, q)$ for even integers p and q, then* $\det D(G) = \frac{-(p+q)^2}{4}$.
(b) *If $G = \theta(2, 2, 2)$, then* $\det D(G) = -16$.
(c) *If $G = \theta(2, 2, q)$ for some odd integer $q > 1$, then* $\det D(G) = q^2 - 5$.
(d) *Otherwise,* $\det D(G) = 0$.

Proof. Items (c) and (d) correspond to Proposition 1 and Proposition 2 of [2], respectively. Item (b) can be computed directly. The proof of case (a) will be divided in the following 2 cases:

Case 1: Let $G = \theta(1, 2, 2k)$, for some $k \geq 1$, with its vertices labeled as in Fig. 2. The distance matrix of $\theta(1, 2, 2k)$ is

$$D(\theta(1, 2, 2k)) = \begin{pmatrix} 0 & v^t \\ v & D(C_{2k+1}) \end{pmatrix},$$

where $D(C_{2k+1})$ is the distance matrix of the cycle induced by the vertices $v_2, \ldots, v_{2k+2}$ and $v^t = (1, 2, \ldots, k, k+1, k, \ldots, 2, 1)$.

From [1], we know that

$$D(C_{2k+1})^{-1} = -2I - C^k - C^{k+1} + \frac{2k+1}{k(k+1)} J, \tag{1}$$

and $\det D(C_{2k+1}) = k(k+1)$, where J is the all ones matrix, with appropriate size, and C is the cyclic permutation matrix of order $2k+1$ having $C_{i,i+1} = 1$ for $i = 1, \ldots, 2k+1$, taking indices modulo $2k+1$. We have that

$$D(\theta(1, 2, 2k))^{-1} = M_1^t M_2 M_1, \tag{2}$$

where

$$M_1 = \begin{pmatrix} 1 & -v^t D(C_{2k+1})^{-1} \\ \mathbf{0} & I \end{pmatrix},$$

$$M_2 = \begin{pmatrix} (-v^t D(C_{2k+1})^{-1} v)^{-1} & \mathbf{0} \\ \mathbf{0} & D(C_{2k+1})^{-1} \end{pmatrix},$$

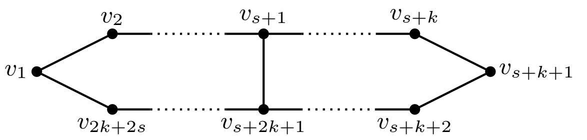

Fig. 3. $\theta(1, 2s, 2k)$.

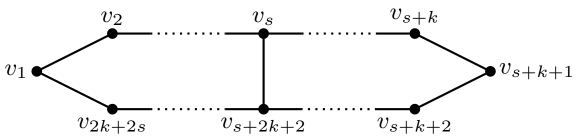

Fig. 4. $\theta(1, 2(s-1), 2(k+1))$.

and

$$\begin{aligned} \det D(\theta(1,2,2k)) &= \det M_2^{-1} = -v^t D(C_{2k+1})^{-1} v \;\; \det D(C_{2k+1}) \\ &= -v^t D(C_{2k+1})^{-1} v \;\, k(k+1). \end{aligned} \tag{3}$$

Now, we will calculate $v^t D(C_{2k+1})^{-1} v$. Using (1) we obtain

$$\begin{aligned} v^t D(C_{2k+1})^{-1} v &= -2v^t v - v^t C^k v - v^t C^{k+1} v + \frac{2k+1}{k(k+1)} v^t J v \\ &= -4 \sum_{i=1}^{k} i^2 - 2(k+1)^2 - 2 \sum_{i=1}^{k} i(k+1-i) \\ &\quad -2 \sum_{i=1}^{k+1} i(k+2-i) + \frac{2k+1}{k(k+1)} (k+1)^4 \\ &= -2 \sum_{i=1}^{k} i(k+1) - 2 \sum_{i=1}^{k+1} i(k+2) + \frac{(2k+1)(k+1)^3}{k} \\ &= -k(k+1)^2 - (k+1)(k+2)^2 + \frac{(2k+1)(k+1)^3}{k} = \frac{k+1}{k}. \end{aligned} \tag{4}$$

Combining this result with (3), we deduce that

$$\det D(\theta(1,2,2k)) = -(k+1)^2 = -\frac{(2k+2)^2}{4} = \frac{-n^2}{4}, \tag{5}$$

with $n = p + q$, where $p = 2$ and $q = 2k$.

Case 2: Let $H = \theta(1, 2s, 2k)$ and $G = \theta(1, 2(s-1), 2(k+1))$, for some $k \geq 2$ and $s \geq 2$, with its vertices labeled as in Fig. 3 and Fig. 4, respectively.

The distance matrices of G and H are

$$D(G) = \begin{pmatrix} P & A^t \\ A & P \end{pmatrix} \quad \text{and} \quad D(H) = \begin{pmatrix} P & B^t \\ B & P \end{pmatrix},$$

where

$$P = \sum_{i=1}^{k+s} \sum_{j=1}^{k+s} |i-j| \; e_i e_j^t, \tag{6}$$

is the distance matrix of P_{k+s} (the path on $k+s$ vertices), and e_i denotes a vector having an entry equal to 1 on the i-th coordinate and 0's in the remaining coordinates. Moreover,

$$\begin{aligned} B^t = & \sum_{j=1}^{k+s} (k+s+1-j) e_1 e_j^t + \sum_{i=2}^{k+s} (k+s+1-i) e_i e_1^t \\ & + \sum_{i=2}^{s+1} \sum_{j=2}^{k+1} (s+k+3-j-i) e_i e_j^t + \sum_{i=s+2}^{s+k} \sum_{j=k+2}^{k+s} (j+i-s-k-1) e_i e_j^t \\ & + \sum_{i=3}^{s+1} s \; e_i e_{i+k-1}^t + \sum_{i=2}^{s+1} \sum_{\substack{j=k+2 \\ j \neq i+k-1}}^{k+s} |r_{2s+1}(1-k+j-i) - s - 1| e_i e_j^t \\ & + \sum_{i=s+2}^{s+k} k \; e_i e_{i-s}^t + \sum_{i=s+2}^{s+k} \sum_{\substack{j=2 \\ j \neq i-s}}^{k+1} |r_{2k+1}(s+j-i) - k - 1| e_i e_j^t \end{aligned} \tag{7}$$

and

$$\begin{aligned} A^t = & \sum_{j=1}^{k+s} (k+s+1-j) e_1 e_j^t + \sum_{i=2}^{k+s} (k+s+1-i) e_i e_1^t \\ & + \sum_{i=2}^{s} \sum_{j=2}^{k+2} (s+k+3-j-i) e_i e_j^t + \sum_{i=s+1}^{s+k} \sum_{j=k+3}^{k+s} (j+i-s-k-1) e_i e_j^t \\ & + \sum_{i=3}^{s} (s-1) \; e_i e_{i+k}^t + \sum_{i=2}^{s} \sum_{\substack{j=k+3 \\ j \neq i+k}}^{k+s} |r_{2s-1}(j-k-i) - s| e_i e_j^t \\ & + \sum_{i=s+1}^{s+k} (k+1) \; e_i e_{i-s+1}^t \end{aligned}$$

$$+\sum_{i=s+1}^{s+k}\sum_{\substack{j=2\\ j\neq i-s+1}}^{k+2}|r_{2k+3}(s+j-i-1)-k-2|e_ie_j^t, \tag{8}$$

where $r_\alpha(\beta)$ represent the remainder when integer β is divided by α.

It is easy to see that P is invertible and

$$\begin{aligned}P^{-1} = &-\frac{k+s-2}{2(k+s-1)}\ e_1e_1^t - \frac{k+s-2}{2(k+s-1)}\ e_{k+s}e_{k+s}^t - \sum_{i=2}^{k+s-1} e_ie_i^t\\ &+\sum_{i=1}^{k+s-1}\frac{1}{2}\ e_ie_{i+1}^t + \sum_{i=2}^{k+s}\frac{1}{2}\ e_ie_{i-1}^t\\ &+\frac{1}{2(k+s-1)}\ e_1e_{k+s}^t + \frac{1}{2(k+s-1)}\ e_{k+s}e_1^t.\end{aligned}$$

We define

$$N := \begin{pmatrix} I & \mathbf{0}\\ (A-MB)P^{-1} & M\end{pmatrix},$$

where

$$M := e_1e_1^t + e_2e_{k+1}^t - e_2e_{k+s}^t + \sum_{i=2}^{k+s} e_ie_{i-1}^t.$$

We claim that

$$D(G) = N\cdot D(H)\cdot N^t. \tag{9}$$

Indeed, it is easy to see that

$$N\cdot D(H)\cdot N^t = \begin{pmatrix} P & A^t\\ A & \widehat{P}\end{pmatrix},$$

where

$$\widehat{P} = A\ P^{-1}\ (A^t - B^tM^t) + (A-MB)\ P^{-1}\ B^t\ M^t + M\ P\ M^t.$$

Hence, it is sufficient to prove that $\widehat{P} = P$. We first compute $M\ P\ M^t$. Since

$$M^t = e_1 e_1^t + e_{k+1} e_2^t - e_{k+s} e_2^t + \sum_{i=2}^{k+s} e_{i-1} e_i^t,$$

$$M\ P = \sum_{j=1}^{k+s} (j-1)\ e_1 e_j^t + \sum_{j=1}^{k+s} (|k+1-j| + 2j - 1 - k - s)\ e_2 e_j^t$$
$$+ \sum_{i=3}^{k+s} \sum_{j=1}^{k+s} |i-1-j|\ e_i e_j^t$$

and

$$\begin{aligned} M\ P\ M^t = &(1-s)\ e_2 e_1^t + (1-s)\ e_1 e_2^t + 4(1-s)\ e_2 e_2^t \\ &+ \sum_{i=3}^{k+s} (i-2)\ e_i e_1^t + \sum_{j=3}^{k+s} (j-2)\ e_1 e_j^t \\ &+ \sum_{j=3}^{k+s} (|k+2-j| + 2j - 3 - k - s)\ e_2 e_j^t \\ &+ \sum_{i=3}^{k+s} (|k+2-i| + 2i - 3 - k - s)\ e_i e_2^t \\ &+ \sum_{i=3}^{k+s} \sum_{j=3}^{k+s} |i-j|\ e_i e_j^t. \end{aligned} \tag{10}$$

We continue obtaining $A^t - B^t M^t$. Multiplying B^t and M^t we obtain

$$B^t M^t = B^t e_1 e_1^t + B^t e_{k+1} e_2^t - B^t e_{k+s} e_2^t + \sum_{i=2}^{k+s} B^t e_{i-1} e_i^t,$$

that is,

$$\begin{aligned} B^t M^t = &\sum_{i=1}^{k+s} (k+s+1-i) e_i e_1^t + (k+2s-1) e_1 e_2^t + \sum_{j=3}^{k+s} (k+s+2-j) e_1 e_j^t \\ &+ \sum_{i=2}^{s} (k+2s+3-3i) e_i e_2^t + \sum_{i=s+1}^{k+s} (k+2-i) e_i e_2^t \\ &+ \sum_{i=2}^{s+1} \sum_{j=3}^{k+2} (s+k+4-j-i) e_i e_j^t + \sum_{i=s+2}^{s+k} \sum_{j=k+3}^{k+s} (j+i-s-k-2) e_i e_j^t \\ &+ \sum_{i=3}^{s} s\ e_i e_{i+k}^t + \sum_{i=2}^{s+1} \sum_{\substack{j=k+3 \\ j \neq i+k}}^{k+s} |r_{2s+1}(j-i-k) - s - 1| e_i e_j^t \end{aligned}$$

$$+\sum_{i=s+2}^{s+k} k\ e_i e_{i-s+1}^t + \sum_{i=s+2}^{s+k} \sum_{\substack{j=3 \\ j\neq i-s+1}}^{k+2} |r_{2k+1}(s+j-i-1)-k-1| e_i e_j^t.$$

From this, we deduce that

$$A^t - B^t M^t = \sum_{i=1}^{s}(2i-2-s)e_i e_2^t + \sum_{i=s+1}^{s+k} s\ e_i e_2^t + \sum_{i=s+1}^{s+k}\sum_{j=3}^{k+s} e_i e_j^t - \sum_{i=1}^{s}\sum_{j=3}^{k+s} e_i e_j^t.$$

It follows that

$$\begin{aligned} P^{-1}(A^t - B^t M^t) &= \Big(-\frac{k+s-2}{2(k+s-1)}\ e_1 e_1^t + \frac{1}{2}\ e_1 e_2^t + \frac{1}{2(k+s-1)}\ e_1 e_{k+s}^t \\ &\quad + \sum_{i=2}^{k+s-1} \frac{1}{2}\ e_i e_{i+1}^t - \sum_{i=2}^{k+s-1} e_i e_i^t + \sum_{i=2}^{k+s-1} \frac{1}{2}\ e_i e_{i-1}^t \\ &\quad + \frac{1}{2(k+s-1)}\ e_{k+s} e_1^t + \frac{1}{2}\ e_{k+s} e_{k+s-1}^t - \frac{k+s-2}{2(k+s-1)}\ e_{k+s} e_{k+s}^t\Big) \\ &\quad \cdot\Big(\sum_{i=1}^{s}(2i-2-s)e_i e_2^t + \sum_{i=s+1}^{s+k} s\ e_i e_2^t + \sum_{i=s+1}^{s+k}\sum_{j=3}^{k+s} e_i e_j^t - \sum_{i=1}^{s}\sum_{j=3}^{k+s} e_i e_j^t\Big) \\ &= e_1 e_2^t + \sum_{i=3}^{k+s} e_s e_i^t - \sum_{i=2}^{k+s} e_{s+1} e_i^t. \end{aligned}$$

Finally, we see that

$$\begin{aligned} AP^{-1}(A^t - B^t M^t) &= A\Big(e_1 e_2^t + \sum_{i=3}^{k+s} e_s e_i^t - \sum_{i=2}^{k+s} e_{s+1} e_i^t\Big) \\ &= s\ e_1 e_2^t + (s-2) e_2 e_2^t \\ &\quad + \sum_{i=3}^{k+1}(s-3) e_i e_2^t + \sum_{i=k+2}^{k+s}(s+2k+1-2i) e_i e_2^t \\ &\quad + \sum_{j=3}^{k+s} e_1 e_j^t - \sum_{i=3}^{k+s}\sum_{j=3}^{k+s} e_i e_j^t, \end{aligned}$$

and

$$\begin{aligned} (A - MB)P^{-1}B^t M^t &= \Big(e_2 e_1^t + \sum_{i=3}^{k+s} e_i e_s^t - \sum_{i=2}^{k+s} e_i e_{s+1}^t\Big) B^t M^t \\ &= s\ e_2 e_1^t + (3s-2) e_2 e_2^t \end{aligned}$$

$$+\sum_{j=3}^{k+1}(s-1)e_2e_j^t+\sum_{j=k+2}^{k+s}(s+2k+3-2j)e_2e_j^t$$
$$+\sum_{i=3}^{k+s}e_ie_1^t+\sum_{i=3}^{k+s}2e_ie_2^t+\sum_{i=3}^{k+s}\sum_{j=3}^{k+s}e_ie_j^t.$$

Thus

$$\begin{aligned}AP^{-1}(A^t-B^tM^t)+(A-MB)P^{-1}B^tM^t &= s\ e_1e_2^t+s\ e_2e_1^t+4(s-1)e_2e_2^t\\ &+\sum_{j=3}^{k+s}e_1e_j^t+\sum_{i=3}^{k+s}e_ie_1^t\\ &+\sum_{i=3}^{k+s}(s+k+1-i-|k+2-i|)e_ie_2^t\\ &+\sum_{j=3}^{k+s}(s+k+1-j-|k+2-j|)e_2e_j^t.\end{aligned}$$

Therefore, by (10), we obtain

$$\widehat{P}=AP^{-1}(A^t-B^tM^t)+(A-MB)P^{-1}B^tM^t+MPM^t=P.$$

This completes the proof of (9). Furthermore, since $\det N\cdot\det N^t=1$, we deduce that

$$\det D(G)=\det D(H).$$

Combining this with (5), using an inductive argument, it follows

$$\det D(\theta(1,2s,2k))=\frac{-n^2}{4},$$

with $n=p+q$, where $p=2s$ and $q=2k$. $\square$

Before computing $\operatorname{cof} D(\theta(l,p,q))$, we need to introduce a new family of bicyclic graphs.

Definition 2. Let l,p,q be positive integers such that at most one of them is equal to one. We define the family of graphs $\Theta'(l,p,q)$ as the set of graphs generated from the $\theta(l,p,q)$-graph by adding a pendant vertex. (see Figs. 5 and 6).

For given l,p,q, since any graph $G\in\Theta'(l,p,q)$ has as blocks the graph $\theta(l,p,q)$ and one edge, it follows from Theorem 2 that the values $\det D(G)$ and $\operatorname{cof} D(G)$ are the same, independent of the vertex of $\theta(l,p,q)$ to which the pendant edge is attached in order to

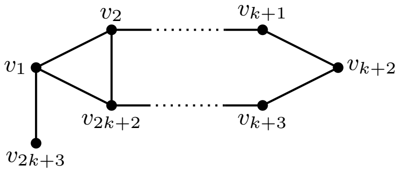

Fig. 5. $\theta'(1, 2, 2k)$.

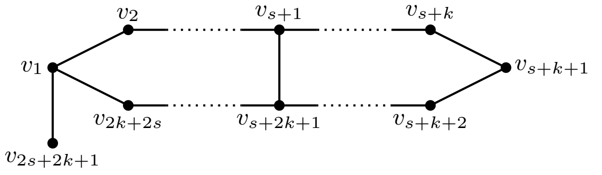

Fig. 6. $\theta'(1, 2s, 2k)$.

obtain G. For this reason, we will use $\theta'(l, p, q)$ for any graph in the family $\Theta'(l, p, q)$. Moreover, from Theorem 2, it follows that

$$\operatorname{cof} D(\theta(l, p, q)) = -2 \det D(\theta(l, p, q)) - \det D(\theta'(l, p, q)).$$

Theorem 4. *Let $G = \theta'(l, p, q) \in \Theta'(l, p, q)$, for integers l, p, q such that at most one of them is 1. Then, the following assertions hold:*

(a) *If $G = \theta'(1, p, q)$ for some even integers p and q, then* $\det D(G) = \frac{(1+p+q)^2-1}{2}$.
(b) *If $G = \theta'(2, 2, 2)$, then* $\det D(G) = 48$.
(c) *If $G = \theta'(2, 2, q)$ for some odd integer $q > 1$, then* $\det D(G) = -2(q^2 + 2q - 9)$.
(d) *Otherwise,* $\det D(G) = 0$.

Proof. Once again, items (c) and (d) can be found in [2] and (b) can be computed directly. The proof of case (a) will be divided in the following 2 cases. All along this proof, $\theta'(l, p, q)$ denotes the graph that arises from $\theta(l, p, q)$ by adding a pendant edge incident precisely to the midpoint of the path of length p joining the two vertices of degree 3 of $\theta(l, p, q)$. Notice that in Figs. 5 and 6 such midpoint is the vertex v_1.

Case 1: Let $G = \theta'(1, 2, 2k)$, for some $k \geq 1$, with its vertices labeled as in Fig. 5. The distance matrix of $\theta'(1, 2, 2k)$ is

$$D(\theta'(1, 2, 2k)) = \begin{pmatrix} 0 & v^t & 1 \\ v & D(C_{2k+1}) & v + \mathbf{1} \\ 1 & v^t + \mathbf{1}^t & 0 \end{pmatrix},$$

where $D(C_{2k+1})$ is the distance matrix of the cycle induced by the vertices $v_2, \ldots, v_{2k+2}$ and $v^t = (1, 2, \ldots, k, k+1, k, \ldots, 2, 1)$. By (2), it follows that

$$\begin{pmatrix} 0 & v^t \\ v & D(C_{2k+1}) \end{pmatrix}^{-1} = M_1^t M_2 M_1.$$

We define

$$M_3 := \begin{pmatrix} I & \mathbf{0} \\ -w^t M_1^t M_2 M_1 & 1 \end{pmatrix},$$

$$M_4 := \begin{pmatrix} M_1^t M_2 M_1 & \mathbf{0} \\ \mathbf{0} & (-w^t M_1^t M_2 M_1 w)^{-1} \end{pmatrix},$$

with $w^t := (1, v^t + \mathbf{1}^t)$. Then,

$$D(G)^{-1} = M_3^t M_4 M_3$$

and

$$\det D(G) = \det M_4^{-1} = -w^t M_1^t M_2 M_1 w \ \det \begin{pmatrix} 0 & v^t \\ v & D(C_{2k+1}) \end{pmatrix}.$$

Combining this result with (5), we conclude that

$$\det D(G) = w^t M_1^t M_2 M_1 w \ (k+1)^2.$$

Now, we will calculate $w^t M_1^t M_2 M_1 w$. From (1), we obtain

$$\begin{aligned} w^t M_1^t M_2 M_1 w &= (0, v^t) M_1^t M_2 M_1 \begin{pmatrix} 0 \\ v \end{pmatrix} + 2\ (0, v^t) M_1^t M_2 M_1 \begin{pmatrix} 1 \\ \mathbf{1} \end{pmatrix} \\ &\quad + (1, \mathbf{1}^t) M_1^t M_2 M_1 \begin{pmatrix} 1 \\ \mathbf{1} \end{pmatrix} \\ &= (0, v^t) \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} + 2\ (1, \mathbf{0}^t) \begin{pmatrix} 1 \\ \mathbf{1} \end{pmatrix} + (1, \mathbf{1}^t) M_1^t M_2 M_1 \begin{pmatrix} 1 \\ \mathbf{1} \end{pmatrix} \\ &= 2 + (1, \mathbf{1}^t) M_1^t M_2 M_1 \begin{pmatrix} 1 \\ \mathbf{1} \end{pmatrix} \\ &= 2 + \frac{v^t D(C_{2k+1})^{-1} v \ \mathbf{1}^t D(C_{2k+1})^{-1} \mathbf{1} - \left(v^t D(C_{2k+1})^{-1} \mathbf{1} - 1 \right)^2}{v^t D(C_{2k+1})^{-1} v}. \end{aligned}$$

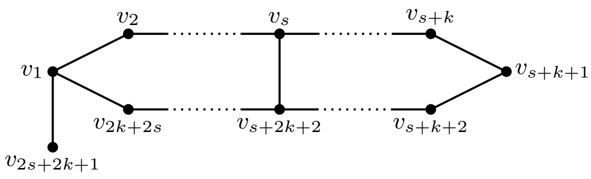

Fig. 7. $\theta'(1, 2(s-1), 2(k+1))$.

By (1), it follows that

$$\begin{aligned} v^t D(C_{2k+1})^{-1}\mathbf{1} &= -2v^t\mathbf{1} - v^t C^k\mathbf{1} - v^t C^{k+1}\mathbf{1} + \frac{2k+1}{k(k+1)} v^t J\mathbf{1} \\ &= -8\sum_{i=1}^{k} i - 4(k+1) + \frac{2k+1}{k(k+1)}(2k+1)(2\sum_{i=1}^{k} i + (k+1)) \\ &= -4(k+1)^2 + \frac{(2k+1)^2}{k}(k+1) = \frac{k+1}{k}, \end{aligned}$$

and

$$\begin{aligned} \mathbf{1}^t D(C_{2k+1})^{-1}\mathbf{1} &= -2\mathbf{1}^t\mathbf{1} - \mathbf{1}^t C^k\mathbf{1} - \mathbf{1}^t C^{k+1}\mathbf{1} + \frac{2k+1}{k(k+1)}\mathbf{1}^t J\mathbf{1} \\ &= -4(2k+1) + \frac{2k+1}{k(k+1)}(2k+1)^2 \\ &= \frac{2k+1}{k(k+1)}. \end{aligned}$$

Thus, by (4), we deduce that

$$w^t M_1^t M_2 M_1 w = 2 + \frac{\frac{k+1}{k}\ \frac{2k+1}{k(k+1)} - \left(\frac{k+1}{k} - 1\right)^2}{\frac{k+1}{k}} = \frac{2k+4}{k+1}.$$

Finally, we obtain

$$\det D(\theta'(1,2,2k)) = (2k+4)(k+1) = -n(n+2m)(-2)^{m-2}, \tag{11}$$

with $n = p + q$ and $m = 1$, where $p = 2$ and $q = 2s$.

Case 2: Let $\widehat{H} = \theta'(1, 2s, 2k)$ and $\widehat{G} = \theta'(1, 2(s-1), 2(k+1), 1)$ be the graphs with its vertices labeled as in Fig. 6 and Fig. 7, respectively, for some $k \geq 2$ and $s \geq 2$.

The distance matrices of $\widehat{G}$ and $\widehat{H}$ are

$$D(\widehat{G}) = \begin{pmatrix} P & A^t & v+\mathbf{1} \\ A & P & w_1+\mathbf{1} \\ v^t+\mathbf{1}^t & w_1^t+\mathbf{1}^t & 0 \end{pmatrix},$$

$$D(\widehat{H}) = \begin{pmatrix} P & B^t & v+\mathbf{1} \\ B & P & w_2+\mathbf{1} \\ v^t+\mathbf{1}^t & w_2^t+\mathbf{1}^t & 0 \end{pmatrix},$$

where P, A and B are the matrices defined in (6), (7) and (8), respectively, v is the first column of P, w_1 is the first column of A and w_2 is the first column of B.

We claim that

$$D(\widehat{G}) = \begin{pmatrix} N & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} D(\widehat{H}) \begin{pmatrix} N^t & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}, \tag{12}$$

where

$$N = \begin{pmatrix} I & \mathbf{0} \\ AP^{-1} - MBP^{-1} & M \end{pmatrix}.$$

Indeed, by (9), it follows that

$$\begin{pmatrix} P & A^t \\ A & P \end{pmatrix} = N \begin{pmatrix} P & B^t \\ B & P \end{pmatrix} N^t,$$

and, hence, it is sufficient to prove that

$$N \begin{pmatrix} v+\mathbf{1} \\ w_2+\mathbf{1} \end{pmatrix} = \begin{pmatrix} v+\mathbf{1} \\ w_1+\mathbf{1} \end{pmatrix}.$$

It is easy to check that

$$N \begin{pmatrix} v+\mathbf{1} \\ w_2+\mathbf{1} \end{pmatrix} = \begin{pmatrix} v+\mathbf{1} \\ (A-MB)P^{-1}(v+\mathbf{1}) + M(w_2+\mathbf{1}) \end{pmatrix}.$$

Since v is the first column of P, we obtain

$$\begin{aligned} (A-MB)P^{-1}v + Mw_2 &= (A-MB)e_1 + Mw_2 \\ &= w_1 - Mw_2 + Mw_2 = w_1. \end{aligned}$$

From the proof of Theorem 3, Case 2, we get that

$$(A-MB)P^{-1} = e_2e_1^t + \sum_{i=3}^{k+s} e_ie_s^t - \sum_{i=2}^{k+s} e_ie_{s+1}^t,$$

and combining this with the definition of M, we conclude that

$$(A - MB)P^{-1}\mathbf{1} + M\mathbf{1} = 0 + \mathbf{1} = \mathbf{1}.$$

This completes the proof of (12). Furthermore, since

$$\det \begin{pmatrix} N & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} \cdot \det \begin{pmatrix} N^t & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} = 1,$$

we deduce that

$$\det D(\widehat{G}) = \det D(\widehat{H}).$$

Combining this with (11) and using an inductive argument, we obtain

$$\det D(\theta'(1, 2s, 2k)) = -n(n + 2m)(-2)^{m-2},$$

with $n = p + q$ and $m = 1$, where $p = 2s$ and $q = 2k$. $\square$

We now consider the case in which a path is attached to a vertex of $\theta(l, p, q)$. We denote by $\theta'_m(l, p, q)$ the graph obtained from $\theta(l, p, q)$ by identifying one vertex of degree three of $\theta(l, p, q)$, with one vertex of degree one of a path of length $m \geq 0$.

The next proposition investigates the determinant of these graphs, when p and q are even.

Proposition 3. *If p and q are even integers, then*

$$\det D(\theta'_m(1, p, q)) = -n(n + 2m)(-2)^{m-2},$$

where $n = p + q$ and $m \geq 0$.

Proof. Let $G_m = \theta'_m(1, p, q)$, $V(G_m) = \{1, \ldots p + q, \ldots, p + q + m\}$ be such that the vertices $\{1, \ldots, p + q\}$ induce $\theta(1, 2s, 2k)$ and the vertices $\{p + q, \ldots, p + q + m\}$ induce P_{m+1}, where $p = 2s$, $q = 2k$ and $m \geq 0$ for some $k \geq 1$ and $s \geq 1$. Arguing as in [6, Theorem 3.2], we obtain

$$\det D(G_m) = -4 \det D(G_{m-1}) - 4 \det D(G_{m-2}),$$

for $m \geq 2$. Combining this identity with the results of Theorem 3, case (a), and Theorem 4, case (a), we deduce that

$$\det D\big(G_m\big) = -n(n + 2m)(-2)^{m-2},$$

where $n = p + q$ and $m \geq 0$. $\square$

As we already know the determinant of any θ-graph and θ'-graph, we obtain the values of $\operatorname{cof} D(G)$, for $G = \theta(l, p, q)$.

Corollary 2. *The following assertions hold:*

- *If* $G = \theta(1, p, q)$ *for some even integers* p *and* q*, then* $\operatorname{cof} D(G) = -(p + q)$.
- *If* $G = \theta(2, 2, 2)$*, then* $\operatorname{cof} D(G) = -16$.
- *If* $G = \theta(2, 2, q)$ *for some odd integer* $q > 1$*, then* $\operatorname{cof} D(G) = 4q - 8$.
- *Otherwise,* $\operatorname{cof} D(G) = 0$.

Remark 1. A graph is said to be *at most bicyclic* if it arises from a tree by the addition of at most two edges. The blocks that are at most bicyclic graphs having at least two vertices are: edge blocks, cycles, and θ-graphs. The values of $\det D(G)$ and $\operatorname{cof} D(G)$ were already known in the first two cases. Now, we have obtained the values of $\det D(G)$ and $\operatorname{cof} D(G)$ for the last case.

From the results above, by applying Theorem 2, we present in the following sequence a formula for $\det D(G)$ for all graphs having at most bicyclic blocks. Notice that this class generalizes the class of cacti (which are graphs having at most unicyclic blocks).

Theorem 5. *Let* G *be a connected graph having blocks at most bicyclic. If* $G = K_1$ *or any block of* G *is an even cycle or a graph* $\theta(l, p, q)$ *with* $\det D(\theta(l, p, q)) = 0$*, then* $\det D(G) = \operatorname{cof} D(G) = 0$*. Otherwise, if the blocks of* G *are:*

- m *edge blocks,*
- c *odd cycles of lengths* $l_1, l_2, \ldots, l_c$*,*
- r *graphs* $\theta(1, p_1, q_1), \theta(1, p_2, q_2), \ldots, \theta(1, p_r, q_r)$ *for even integers* $p_1, q_1, \ldots, p_r, q_r$*,*
- s *graphs* $\theta(2, 2, 2)$*, and*
- t *graphs* $\theta(2, 2, q_1), \theta(2, 2, q_2), \ldots, \theta(2, 2, q_t)$ *for odd integers* $q_1, q_2, \ldots, q_t > 1$*,*

then

$$\det D(G) = \left(\frac{m}{2} + \sum_{h=1}^{c} \frac{l_h^2 - 1}{4l_h} + \sum_{i=1}^{r} \frac{p_i + q_i}{4} + s + \sum_{j=1}^{t} \frac{q_j^2 - 5}{4q_j - 8} \right) \operatorname{cof} D(G),$$

where

$$\operatorname{cof} D(G) = (-2)^m (-1)^r (-16)^s \left(\prod_{h=1}^{c} l_h \right) \left(\prod_{i=1}^{r} (p_i + q_i) \right) \prod_{j=1}^{t} (4q_j - 8).$$

Declaration of competing interest

None declared.

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