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Convex p-partitions of bipartite graphs

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ABSTRACT

A set of vertices X of a graph G is *convex* if no shortest path between two vertices in X contains a vertex outside X. We prove that for fixed $p \geq 1$, all partitions of the vertex set of a bipartite graph into p convex sets can be found in polynomial time.

1. Introduction

Given a graph $G = (V, E)$, a set X of vertices is called *convex* if $G[X]$, the graph induced by X, contains all shortest paths between any two of its vertices. All graphs here are undirected and simple. The notion probably first appeared in [8], see also [10], and later became also known as geodesic convexity, or d-convexity, in order to distinguish it from different notions of convexity in graphs and other combinatorial structures (see [7] for an early overview). The book [11] gives an up-to-date survey of results on convexity in graphs.

One of the approaches to convexity in graphs comes from the viewpoint of computational complexity. Clearly, by computing the distances between all pairs, one can decide in polynomial time if a given set of vertices is convex. To determine the size of a largest convex set not covering the whole graph, however, is an NP-complete problem, even for bipartite graphs, albeit linear for cographs [5]. The same phenomenon occurs (NP-completeness even for bipartite graphs, but linearity for cographs) if we wish to determine related invariants such as the hull number and the geodetic number of a graph [1,4,6].

We focus here on the notion of a *convex p-partition* of a graph, that is, a partition of the vertex set into p convex sets. For instance, any graph on n vertices containing a matching of size m has a convex $(n-m)$-partition, and trivially, any graph has a convex 1-partition. Deciding whether a graph has a convex p-partition, for fixed $p \geq 2$, is NP-complete for arbitrary graphs, and linear time solvable for cographs [2]. Also, any connected chordal graph has at least one convex p-partition for each $p \geq 1$ [2].

In view of the above described panorama, it was conjectured in [11] that also for bipartite graphs, it should be NP-complete to decide whether they have a convex p-partition. We show that, for any fixed $p \geq 1$, this is not the case. More

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precisely, we prove that for $p \geq 1$, all convex p-partitions of a bipartite graph can be enumerated in polynomial time. This extends a recent result of Glantz and Meyerhenke [9], who prove the same for the case $p = 2$. They also showed that all convex 2-partitions of a planar graph can be found in polynomial time.

2. Bipartite graphs with convex p-partitions

We start by reproving the result for bipartite graphs from [9] in a slightly different way. At the same time, this will serve as a base for the general case. We denote the distance between two vertices u and v in a graph G by $d_G(u, v)$, defined as the length of a shortest path between u and v. It is known that for a given u, the set of all distances $d(u, v)$, for $v \in V$, can be computed in linear time [12].

Lemma 1. *Given a convex set C in a connected bipartite graph G, and an edge uv with $u \in C$, $v \notin C$ we have that $d_G(u', u) < d_G(u', v)$, for each $u' \in C$.*

Proof. Suppose otherwise. Observe that since G is bipartite, $d_G(u', u) \neq d_G(u', v)$, and thus we may assume $d_G(u', u) > d_G(u', v)$. Then there is a shortest path P from u' to v not containing u. Extending P to u through the edge vu, gives a shortest path from u' to u, a contradiction, as u and u' lie in the convex set C, but $v \notin C$. $\square$

Let $e = uv$ be an edge of G and denote by X_{uv} the set of vertices that are closer to u than to v. If G is a connected bipartite graph, then V is the disjoint union $X_{uv} \cup X_{vu}$. From Lemma 1 we get the following corollaries.

Corollary 2. *Let uv be an edge of a connected bipartite graph G. If C is a convex set containing u and not containing v, then $C \subseteq X_{uv}$.*

Corollary 3. *Let $G = (V, E)$ be a connected bipartite graph, with a partition of V into convex sets X_1, X_2. Let $uv \in E$, with $u \in X_1$ and $v \in X_2$. Then $X_1 \subseteq X_{uv}$ and $X_2 \subseteq X_{vu}$ which, as $V = X_{uv} \cup X_{vu}$, implies that $X_1 = X_{uv}$ and $X_2 = X_{vu}$.*

From the previous corollary it is direct that there are at most $|E|$ convex 2-partitions and, as a consequence, we can enumerate all convex 2-partitions in polynomial time.

Proposition 4. *We can enumerate in polynomial time all convex 2-partitions of a connected bipartite graph.*

We now prove that for fixed $p \geq 3$, we can enumerate in polynomial time all convex p-partitions of a connected bipartite graph. In order to do so, we extend the idea present in Corollary 3.

We write $[p]$ for the set $\{1, \ldots, p\}$. For a set F of edges, let $V(F)$ denote the set of all endvertices of edges of F.

Given a convex p-partition $\mathcal{X} = \{X_1, X_2, \ldots, X_p\}$ of a graph $G = (V, E)$, we call a pair (F, ϕ) an *$\mathcal{X}$-skeleton*, if $F \subseteq E$ and $\phi : V(F) \to [p]$ satisfy the following:

- all edges of F go between distinct parts of $\mathcal{X}$;
- if there is at least one edge in E between X_i and X_j, then there is exactly one edge of F between X_i and X_j;
- $\phi(v) = i$ if and only if $v \in X_i$.

Note that the first two conditions might be equivalently expressed by saying that after contracting the sets X_i and deleting all remaining edges that are not in F, we are left with a (simple) graph $H_{(F,\phi)}$ whose edges represent the edges of G that cross the partition. The last condition says ϕ assigns the same color to all vertices of $V(F)$ that become identified in $H_{(F,\phi)}$.

Note that for a connected graph G the second condition implies that $V(F) \cap X_j \neq \emptyset$, for each $j \in [p]$. Then, the third condition implies that ϕ is a surjective function.

Given a set of edges F we say that a function $\phi : V(F) \to [p]$ is a *p-coloring* of F if it is surjective and for each $vw \in F$, $\phi(v) \neq \phi(w)$.

We shall prove that given a graph $G = (V, E)$, $F \subseteq E$ and ϕ a p-coloring of F, we can decide in linear time whether (F, ϕ) is the $\mathcal{X}$-skeleton of a convex p-partition $\mathcal{X} = \{X_1, X_2, \ldots, X_p\}$ of G.

To this end, we use the following two criteria which follow from Corollary 2 and the definition of a convex set, respectively.

1. For each $i \in [p]$ and for each edge $vw \in F$ with $\phi(w) = i$, if a vertex $u \in X_{vw}$ then $u \notin X_i$.
2. For each $i \in [p]$, for any three distinct vertices u, v, w with $w \in V(F)$ and $d_G(u, w) = d_G(u, v) + d_G(v, w)$, if $v \notin X_i$ and $w \in X_i$, then $u \notin X_i$.

The algorithm described in Algorithm 1 has three steps. It starts with considering for each part of the convex partition the whole set of vertices. In a second step, it eliminates from each part X_i those vertices indicated by the first criterion. For each $vw \in F$ we can compute in linear time the set X_{vw}, and thus, we can check in constant time whether $u \in X_{vw}$.

Algorithm 1 p-is-skeleton.

Require: A graph $G = (V, E)$ and (F, ϕ), $F \subseteq E$, ϕ a p-coloring of F.
Return: $\mathcal{X}$ – a convex p-partition of G having (F, ϕ) as skeleton, if it exists.
For all $i \in [p]$
  $X_i^1 \leftarrow V$;
  For all $vw \in F$ with $\phi(w) = i$
    For all $u \in X_{vw}$
      $X_i^1 \leftarrow X_i^1 \setminus \{u\}$;
      If $X_i^1 = \emptyset$ **then return** False.
For all $i \in [p]$
  $X_i^2 \leftarrow X_i^1$;
  For all $w \in V(F)$ with $\phi(w) = i$
    For all $u \in X_i^1$ s.t. $\exists v \in V \setminus X_i^1$ with $d_G(u, v) + d_G(v, w) = d_G(u, w)$
      $X_i^2 \leftarrow X_i^1 \setminus \{u\}$;
      If $X_i^2 = \emptyset$ **then return** False.
For all $i \in [p]$
      If X_i^2 is not convex **then return** False.
Return: $\mathcal{X} = \{X_1^2, \ldots, X_p^2\}$

Therefore, this part takes linear time. Finally, in the third step, the algorithm eliminates all vertices indicated by the second criterion from the parts obtained in the previous step. As before, for each $w \in V(F)$ we can compute, in linear time, the distance from w to all the vertices, and during the same process, we can already eliminate the vertices that are as in the second criterion. Hence, Algorithm 1 runs in linear time.

The correctness of this algorithm is proved in the next result. However, in the proof, instead of working with parts X_i, we associate to each vertex the set of indices of the parts to which it belongs. Initially this set is $[p]$, in the second step we erase for these sets all the indices indicated by the first criterion, and in the third step we erase from the remaining indices those indicated by the second criterion.

For each pair of vertices u and w in a graph G we define $I[u, w]$ as the set of vertices in shortest paths between u and w. Then, $v \in I[u, w]$ if and only if $d_G(u, w) = d_G(u, v) + d_G(v, w)$.

Theorem 5. *Let $G = (V, E)$ be a connected bipartite graph, let $F \subseteq E$ and let $\phi : V(F) \to [p]$. If G has a convex p-partition with skeleton (F, ϕ), then this partition is unique. We can find such partition, or show it does not exist, in polynomial time.*

Proof. Define lists $L(u)$ for each vertex $u \in V$ by setting

$$L(u) := [p] - \{\phi(w) \colon u \in X_{vw} \text{ for some } vw \in F\}.$$

The idea behind the lists $L(u)$ is that they do not contain colors corresponding to partition sets u cannot belong to, as explained in more detail above. Restricting these lists even more, we define, for each vertex u,

$$L'(u) := L(u) - \{\phi(w) \colon w \in V(F) \text{ and } \phi(w) \notin L(v) \text{ for some } v \in I[u, w]\}.$$

We will prove that if G has a convex p-partition $\mathcal{X} = \{X_1, \ldots, X_p\}$ with skeleton (F, ϕ), then, for each $i \in [p]$,

$$L'(u) = \{i\} \text{ for every } u \in X_i. \tag{1}$$

We first observe that

$$i \in L(u) \text{ for every } u \in X_i. \tag{2}$$

Otherwise, there are $u \in X_i$ and $vw \in F$ such that $\phi(w) = i$ and $u \in X_{vw}$. Hence, $u, w \in X_i$ and $v \in I[u, w]$. Since X_i is convex, $v \in X_i$, contradicting the fact that the edge vw of F must join distinct parts of $\mathcal{X}$. This contradiction proves (2).

Moreover,

$$i \in L'(u) \text{ for every } u \in X_i. \tag{3}$$

Otherwise, by (2), there are $u \in X_i$, $w \in V(F)$ and $v \in I[u, w]$ such that $\phi(w) = i$ and $i \notin L(v)$. Now, on the one hand, since $u, w \in X_i$ and $v \in I[u, w]$, the convexity of X_i implies that $v \in X_i$. On the other hand, since $i \notin L(v)$, we know by (2) that $v \notin X_i$. This contradiction proves (3).

Next, we now show that, for each $j \in [p]$,

$$\text{if } v'w' \in E, \text{ with } w' \in X_j \text{ and } v' \notin X_j, \text{ then } j \notin L(v'). \tag{4}$$

This is immediate if $v'w' \in F$, by the definition of $L(v')$. Otherwise, there is $vw \in F$ such that $w \in X_j$, and $v, v' \in X_i$ for some $i \neq j$. Lemma 1 applied to the convex set X_i and the edge vw yields that $d(v', v) < d(v', w)$; i.e., $v' \in X_{vw}$. Thus, the definition of $L(v')$ gives that $j \notin L(v')$, proving (4).

We now prove (1). Consider $u \in X_i$ and $j \in [p] - \{i\}$. Let $w \in V(F) \cap X_j$ (as G is connected, this set is non-empty) and let P be a shortest path between u and w. By construction, P has some edge vw' such that $v \notin X_j$ and $w' \in X_j$. By (4), we have that $j \notin L(v)$. As $v \in I[u, w]$, and as $\phi(w) = j$, the definition of $L'(u)$ implies that $j \notin L'(u)$. This completes the proof of (1).

Therefore, a convex p-partition with skeleton (F, ϕ) exists if and only if the following conditions hold: (i) $|L'(u)| = 1$ for each vertex u of G; (ii) the parts of the corresponding partition are convex. The time needed to find a convex p-partition with skeleton (F, ϕ) is dominated by the time needed to compute the set of distances $d_G(w, u)$ for each $w \in V(F)$ and each $u \in V$ which is linear for each set F. Indeed, for each $w \in V(F)$ we can make a breadth first search starting at w and delete $\phi(w)$ from $L(u)$, for those u in X_{vw}. Similarly, we can construct lists $L'(u)$, by running for each $w \in V(F)$, a breath first search starting at w during which we delete $\phi(w)$ from $L(u)$ of all the descendent u of a vertex v for which $\phi(w) \notin L(v)$. $\square$

When given a connected bipartite graph G and an integer p, we can decide whether G has a convex p-partition as follows. We first *guess* a *candidate* skeleton (F, ϕ) with ϕ a p-coloring of F, and then, by using Theorem 5, we compute in linear time the unique (if any) partition $\{X_1, \ldots, X_p\}$ associated to (F, ϕ). The choices for (F, ϕ) are bounded from above by a function that depends only on p. In fact, if (F, ϕ) is a skeleton of some partition, then it must satisfy the following properties.

- The size of F satisfies $|F| \in \{p-1, \ldots, \binom{p}{2}\}$.
- The function ϕ is surjective.
- Identifying all vertices $v \in V(F)$ of the same color under ϕ yields a connected simple graph.

From the first condition we know that there are at most

$$\binom{\binom{n}{2}}{|F|} \leq \binom{n}{2}^{|F|}$$

choices for the set F. Hence, there are at most $\binom{p}{2}\binom{n}{2}^{\binom{p}{2}}$ choices in total.

From the second condition we know that there are roughly $\binom{p^2}{|F|} \leq p^{2p}$ functions ϕ. Since the problem of determining the convex p-partitions of a graph can be reduced in polynomial time to computing the convex p'-partitions of its components for $p' \in \{1, \ldots, p\}$ [2,3], we conclude the following.

Corollary 6. *For each fixed $p \geq 1$, all convex p-partitions of a bipartite graph can be enumerated in polynomial time.*

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