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Forbidden induced subgraphs of normal Helly circular-arc graphs: Characterization and detection[☆]

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ABSTRACT

A normal Helly circular-arc graph is the intersection graph of a set of arcs on a circle of which no three or less arcs cover the whole circle. Lin et al. (2013) characterized circular-arc graphs that are not normal Helly circular-arc graphs, and used them to develop the first recognition algorithm for this graph class. As open problems, they ask for the forbidden subgraph characterization and a direct recognition algorithm for normal Helly circular-arc graphs, both of which are resolved by the current paper. Moreover, when the input is not a normal Helly circular-arc graph, our recognition algorithm finds in linear time a minimal forbidden induced subgraph as a certificate. Our approach yields also a considerably simpler algorithm for the certifying recognition of proper Helly circular-arc graphs, a subclass of normal Helly circular-arc graphs.

1. Introduction

A graph is a *circular-arc graph* if its vertices can be assigned to arcs on a circle such that two vertices are adjacent if and only if their corresponding arcs intersect. Such a set of arcs is called a *circular-arc model* for this graph. If some point on the circle is not covered by any arc in the model, then the graph can also be represented by a set of intervals on the real line. This set of intervals is called an *interval model*, and the graph is an *interval graph*. The intersection graph of a set of subtrees of a tree is called a *chordal graph*. Circular-arc graphs, interval graphs, and chordal graphs are three of the most famous intersection graph classes, and have been studied intensively for more than half century. In contrast to interval graphs and chordal graphs, however, our understanding of circular-arc graphs is far limited, and to date some fundamental problems remain unsolved.

One fundamental combinatorial problem on a graph class is its characterization by forbidden (induced) subgraphs. For example, the forbidden induced subgraphs of *chordal graphs* are holes (i.e., induced cycles of length at least four). Lekkerkerker and Boland [13] showed in 1962 that the forbidden induced subgraphs of interval graphs include holes and graphs in Fig. 1. In contrast, the characterization of circular-arc graphs by forbidden induced subgraphs remains a notorious open problem since it was first asked by Hadwiger et al. [7] in 1964, though previous attempts did have partial success, which is mainly on subclasses of circular-arc graphs. For example, Tucker [23] characterized unit circular-arc graphs (i.e., a graph with a circular-arc model where every arc has the same length) and proper circular-arc graphs (i.e., a graph with a circular-arc model where

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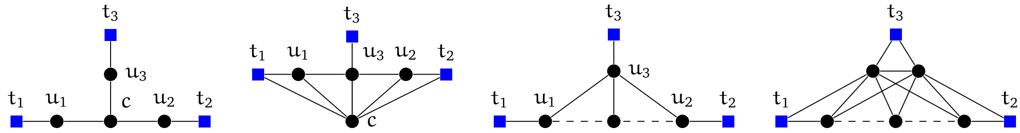
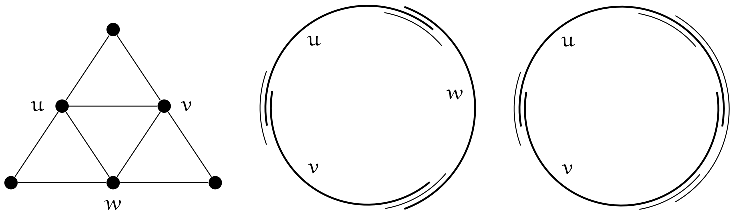


(a) Long claw. (b) Whipping top. (c) †. (d) ‡.

Fig. 1. Chordal minimal forbidden induced graphs.



(a) Tent (a ‡ on 6 vertices). (b) A normal model. (c) A Helly model.

Fig. 2. Tent and its circular-arc models. The arcs ($\{u, v, w\}$) invalidating the Helly property in (b) and the arcs ($\{u, v\}$) invalidating the normal property in (c) are marked as thick.

no arc properly contains another). We refer to the surveys of Lin and Soulignac [17] and of Durán et al. [4] for recent results in this line. The recent breakthrough of Francis et al. [5] may shed some light on the final resolution of this problem.

One fundamental algorithmic problem on a graph class is its recognition, i.e., to efficiently decide whether a given graph belongs to this class or not. For intersection graph classes, all recognition algorithms known to the authors provide an intersection model or some equivalent structure when the membership is asserted. Most of them, on the other hand, simply return "NO" for non-membership, while one might also want a verifiable *certificate* for some reason [19]. A recognition algorithm is *certifying* if it provides both positive and negative certificates. There are different forms of negative certificates, while a minimal forbidden induced subgraph is arguably the simplest and most preferable of them [8]. For example, it is long known that a hole can be detected from a non-chordal graph in linear time [22]. Very recently, Lindzey and McConnell [14] reported a linear-time algorithm that detects a subgraph in Fig. 1 from a chordal non-interval graph. They together make a linear-time certifying algorithm for the recognition of interval graphs. On the other hand, although a circular-arc model for a circular-arc graph can be produced in linear time [18], it remains a challenging open problem to find a negative certificate for a non-circular-arc graph in the same time.

The complication of circular-arc graphs may be attributed to two special intersection patterns of circular-arc models that are not possible in interval models. The first is two arcs intersecting in their both ends, and a circular-arc model is called *normal* if no such pair exists. The second is a set of arcs intersecting pairwise but containing no common point, and a circular-arc model is called *Helly* if no such set exists. Normal and Helly circular-arc models are precisely those without three or less arcs covering the whole circle [20,15]. A graph that admits such a model is called a *normal Helly circular-arc graph*. Clearly, all interval graphs are normal Helly circular-arc graphs; indeed, one may verify that all normal Helly circular-arc graphs that are chordal are interval graphs.

A word of caution is worth on the definition of normal Helly circular-arc graphs. One graph might admit both a normal circular-arc model and a Helly circular-arc model but not a circular-arc model that is both normal and Helly. See Fig. 2 for an example. One may want to verify that arranging a normal and Helly circular-arc model for a tent (i.e., a ‡ on 6 vertices) is out of the question. This example convinces us that the class of normal Helly circular-arc graphs is *not* equivalent to the intersection of the class of normal circular-arc graphs and the class of Helly circular-arc graphs, but a proper subset of it.

Let us mention some previous work related to normal Helly circular-arc graphs. The algorithm of Tucker [24] colors a normal Helly circular-arc graph using at most $3\omega/2$ colors, where ω denotes the size of its maximum cliques. Note that by the Helly property, ω is equivalent to the maximum number of arcs covering a single point on the circle. This is tight as any odd hole, which has $\omega = 2$ and needs at least three colors, is a normal Helly circular-arc graph. In the study of convergence of circular-arc graphs under the clique operator, Lin et al. [16] observed that normal Helly circular-arc graphs arose naturally. They [15] then undertook a systematic study of normal Helly circular-arc graphs as well as its subclass. Their results include a partial characterization of normal Helly circular-arc graphs by forbidden induced subgraphs (more specifically, those restricted to Helly circular-arc graphs), and a linear-time recognition algorithm (by calling a recognition algorithm for circular-arc graphs). As open problems, they ask for determining the remaining minimal forbidden induced subgraphs, and designing a direct recognition algorithm, both of which are resolved by the current paper.

The first main result of this paper is a complete characterization of normal Helly circular-arc graphs by forbidden induced subgraphs. A wheel (resp., C^*) comprises a hole and another vertex completely adjacent (resp., nonadjacent) to the hole.

Theorem 1.1. *A graph is a normal Helly circular-arc graph if and only if it contains no C^*, wheel, or any graph depicted in* Figs. 1 and 3 *as an induced subgraph.*

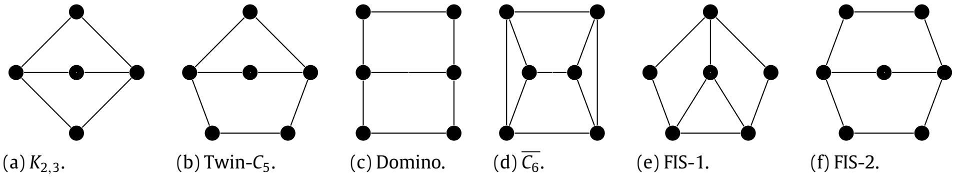


Fig. 3. Non-chordal and finite minimal forbidden induced graphs.

Let $\mathcal{F}$ denote the set of C^*, wheel, and all graphs depicted in Figs. 1 and 3. First, a C^* is not a circular-arc graph, while a wheel cannot be arranged without three or less arcs covering the circle. Second, every graph in Fig. 1 is a chordal but non-interval graph, and thus cannot be a normal Helly circular-arc graph. Third, every graph in Fig. 3 has only a small number of vertices and can be easily checked. Therefore, each graph in $\mathcal{F}$ is not a normal Helly circular-arc graph, and to prove Theorem 1.1, it suffices to show that a graph containing none of them is a normal Helly circular-arc graph. That fact was actually proved in [6], but the proof given there does not imply a linear-time procedure to find the corresponding forbidden induced subgraphs when the graph is not a normal Helly circular-arc graph. Such a procedure was subsequently discovered by the first author in [2], and we decide to merge our work into a joint paper.

It is known that if a normal Helly circular-arc graph G is not chordal, then every circular-arc model for G has to be normal and Helly [20,15]. This observation inspires us to recognize normal Helly circular-arc graphs as follows. If the input graph is chordal, it suffices to check whether it is an interval graph. Otherwise, we try to build a circular-arc model for it, and if we succeed, verify whether the model is normal and Helly. Lin et al. [15] showed that this approach can be implemented in linear time. Moreover, if there exists a set of at most three arcs covering the circle, then their algorithm returns it as a certificate. This algorithm, albeit conceptually simple, suffers from twofold weakness. First, it needs to call some recognition algorithm for circular-arc graphs, while all known ones are extremely complicated. Second, it is very unlikely to deliver a negative certificate in general.

The second main result of this paper is the following recognition algorithm. We use $n := |V(G)|$ and $m := |E(G)|$ throughout.

Theorem 1.2. *There is an $O(n+m)$-time algorithm that given a graph G, either constructs a normal and Helly circular-arc model for G, or finds a subgraph of G that is in $\mathcal{F}$.*

Since the algorithm of Theorem 1.2 always finds a subgraph in $\mathcal{F}$ when the input graph is not a normal Helly circular-arc graph, Theorem 1.1 follows from the correctness proof as a corollary. Let us briefly discuss the basic idea behind the way we deal with a non-chordal graph G. If G is a normal Helly circular-arc graph, then for any vertex v of G, both $N[v]$ and its complement induce nonempty interval subgraphs. The main technical difficulty is how to combine interval models for them to make a circular-arc model for G. For this purpose we build an auxiliary graph $\mho(G)$ by taking two identical copies of $N[v]$ and appending them to the "two ends" of $G - N[v]$ respectively. The shape of symbol $\mho$ is a good hint for understanding the structure of the auxiliary graph. We show that $\mho(G)$ is an interval graph and more importantly, a circular-arc model for G can be produced from an interval model for $\mho(G)$. On the other hand, if G is not a normal Helly circular-arc graph, then either the construction of $\mho(G)$ fails, or it is not an interval graph. In the final case we use the following procedure to obtain a minimal forbidden induced subgraph of G.

Theorem 1.3. *Given a minimal non-interval induced subgraph of $\mho(G)$, we can in $O(n+m)$ time find a subgraph of G that is in $\mathcal{F}$.*

A circular-arc or interval model is *proper* if no arc or interval in it properly contains another arc or interval. A graph is a *proper interval/circular-arc graph* if has a proper interval/arc model respectively. A graph is a *proper Helly circular-arc graph* if it has a circular-arc model that is both proper and Helly. One should again be warned that a circular-arc graph might have a proper model and a Helly model but no model that is both proper and Helly, e.g., the model given in Fig. 2(b) is proper. It is known that proper Helly circular-arc graphs are exactly normal Helly circular-arc graphs that are claw-free [15]. Therefore, our recognition algorithm for normal Helly circular-arc graphs can be turned to a recognition algorithm for proper Helly circular-arc graphs as follows. It is trivial when the input graph is not a normal Helly circular-arc graph. Otherwise, we have a normal and Helly circular-arc model, and it suffices to apply the algorithm of Kaplan and Nussbaum [10], which either makes the model proper or finds a claw. This is already better than the previous algorithm based on recognizing (proper) circular-arc graphs [15]. Actually, we can do even better. We observe that if G is a proper Helly circular-arc graph, then the auxiliary graph $\mho(G)$ is a proper interval graph, which admits an even simpler certifying recognition algorithm [3]. Similar as Theorem 1.3, a minimal non-proper-interval subgraph of $\mho(G)$ can be translated to a negative certificate of G. Therefore, we provide an alternative and considerably simpler proof for the following.

Theorem 1.4 (*[15]*)**.** *There is an $O(n+m)$-time algorithm that given a graph G, either constructs a proper and Helly circular-arc model for G, or finds a minimal subgraph of G that is not a proper Helly circular-arc graph.*

The crucial idea behind our certifying algorithms is a novel correlation between normal Helly circular-arc graphs and interval graphs, which can be efficiently used for algorithmic purpose. This correlation was originally observed in the detection of small non-interval subgraphs [1], which used a similar definition of the auxiliary graph and pertinent observations. However, the main structures and the procedures for their detection divert completely. For example, the most common forbidden induced subgraphs in [1] are C_4's and C_5's, which, however, are allowed in normal Helly circular-arc graphs. This means that the interaction between $N[v]$ and $G - N[v]$ in the current paper are far more subtle.

A similar observation, working on interval subgraph $G - N[v]$ and then adding back $N[v]$, had been used by Hsu and Spinrad [9] to obtain the first linear-time algorithm for finding a maximum independent set in a circular-arc graph. Interestingly, their algorithm picks a vertex with the minimum degree as the special vertex, while in our case a vertex with a larger degree is preferred. It is worth mentioning that back then there was no known linear-time recognition algorithm for circular-arc graphs. Thus, they could not assume the possession of a circular-arc model for the graph, and their algorithm has to take into consideration that the input might not be a circular-arc graph. This became unnecessary after the linear-time recognition was finally discovered, first by McConnell [18] and then by Kaplan and Nussbaum [11]. However, we might want to avoid calling these recognition algorithms for some subclasses of circular-arc graphs.

2. The recognition algorithm

This paper will be only concerned with undirected and simple graphs. All graphs are stored as adjacency lists. We use the customary notation $v \in G$ to mean $v \in V(G)$, and $u \sim v$ to mean $uv \in E(G)$. The *degree* of a vertex v is defined by $d(v) := |N(v)|$, where the *neighborhood* $N(v)$ of v comprises all vertices u such that $u \sim v$. The *closed neighborhood* of v is defined by $N[v] := N(v) \cup \{v\}$. For a vertex set U, its closed neighborhood and neighborhood are defined by $N[U] := \bigcup_{v \in U} N[v]$ and $N(U) := N[U] \backslash U$ respectively. The length of a cycle or a path is the number of edges it contains. For $\ell \geq 4$, we use C_ℓ to denote an hole on ℓ vertices; likewise, a C^* and a wheel with a hole C_ℓ are denoted by a C^*_ℓ and a W_ℓ respectively. Exclusively concerned with induced subgraphs, we will abuse notation by using the same symbol to denote a subset of vertices and the subgraph induced by it.

Consider a circular-arc model $\mathcal{A}$ that is both normal and Helly. It is trivially equivalent to an interval model if some point of the circle is not covered by any arc in $\mathcal{A}$. Otherwise, we can find an inclusion-wise minimal set X of arcs that cover the entire circle. It consists of at least four arcs [20,15] and thus represents a hole. Therefore, a normal Helly circular-arc graph G is chordal if and only if it is an interval graph, for which it suffices to call the algorithms of [12,14]. We are hence focused on graphs that are not chordal.

Proposition 2.1. *Let H be a hole of a circular-arc graph G. In any circular-arc model for G, the union of arcs for H covers the whole circle.*

As a result, every vertex should have neighbors on any hole H. We use $N_H[v]$ as a shorthand for $N[v] \cap H$, regardless of whether $v \in H$ or not. Using circular-arc models one can verify that every hole of a circular-arc graph has the following properties:

(P1') For every vertex v, the set $N_H[v]$ is nonempty and its vertices are consecutive on H.
(P2') For any pair of adjacent vertices u, v, the intersection of $N_H[u]$ and $N_H[v]$ is nonempty.

As we will see shortly, stronger versions of them are satisfied by every hole of a normal Helly circular-arc graph.

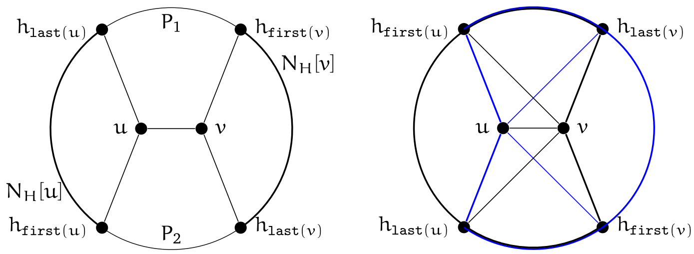
(P1) For every vertex v, the set $N_H[v]$ of vertices induces a nonempty sub-path of H.
(P2) For any pair of adjacent vertices u, v, either one of $N_H[u]$ and $N_H[v]$ is a subset of the other, or precisely one end of (the sub-path induced by) $N_H[u]$ is contained in $N_H[v]$.

Property P1 strengthens P1' by excluding the case $N_H[v] = H$ (i.e., $H \subseteq N[v]$), when H and v make a wheel. Property P2 strengthens P2' by excluding the case where both ends of $N_H[u]$ are in $N_H[v]$ but $N_H[u] \not\subseteq N_H[v]$; note that if the second case of P2 holds, then precisely one end of $N_H[v]$ is contained in $N_H[u]$. A hole H is denoted by $h_0 h_1 \cdots h_{|H|-1}$, and the indices of its vertices should be understood as modulo $|H|$, e.g., $h_{-1} = h_{|H|-1}$. We designate the ordering $h_0, h_1, h_2, \ldots$ of traversing H as *clockwise*, and the other *counterclockwise*.

Lemma 2.2. *Given a hole H, we can in $O(n + m)$ time either determine that H satisfies property* P1*, or produce a subgraph of G that is in $\mathcal{F}$. The sub-paths $N_H[v]$ for all vertices v can be found in the same time if property* P1 *is satisfied.*

Proof. Since $N_H[h_i] = \{h_{i-1}, h_i, h_{i+1}\}$ for every $0 \leq i < |H|$, vertices on H will not concern us. For each $v \in V(G)\backslash H$, we mark vertices in $N_H[v]$ "a neighbor of v", and count the number. If it is 0 or $|H|$, then we return H and v as a C^* or wheel respectively. Otherwise, from any neighbor of v on H, we traverse clockwise and counterclockwise till the first vertices not marked "a neighbor of v". If the total number of visited neighbors of v during this traversal is the same as $|N_H[v]|$, then we have the sub-path $N_H[v]$, and v passes the test. Either all vertices have passed the test, which means that H satisfies property P1, or we construct the subgraph of G that is in $\mathcal{F}$ as follows.

Let v be the vertex that failed the test. Let $\{h_{p_2}, h_{p_2+1}, \ldots, h_{p_3}\}$ be the set of neighbors of v visited in the test; then $p_3 - p_2 + 1 < |N_H[v]|$, and neither of h_{p_2-1} and h_{p_3+1} is adjacent to v. We traverse H counterclockwise from h_{p_2-1} and



(a) A pair of vertices failing (i).

(b) A pair of vertices failing (ii).

Fig. 4. The neighborhoods of a pair of adjacent vertices on H (Lemma 2.3).

clockwise from h_{p_3+1} till the first neighbor(s) of v on H; let them be h_{p_1} and h_{p_4} respectively. Note that they possibly refer to the same vertex, and this fact is irrelevant in our construction below. Now we have two nontrivial sub-paths, $h_{p_1}h_{p_1+1}\dots h_{p_2}$ and $h_{p_3}h_{p_3+1}\dots h_{p_4}$, of H such that v is adjacent to their ends but none of their inner vertices. Without loss of generality, assume $0 \le p_1 < p_2 \le p_3 < p_4 \le |H|$; then $p_2 - p_1 > 1$ and $p_4 - p_3 > 1$.

If $p_2 - p_1 > 3$, then we return $vh_{p_3}h_{p_3+1}\dots h_{p_4}$ and h_{p_1+2} as a C^*. Likewise, if $v \not\sim h_\ell$ for some ℓ with $p_2 + 1 < \ell < p_1 - 1 + |H|$, then we return $vh_{p_1}h_{p_1+1}\dots h_{p_2}$ and h_ℓ as a C^*; note that this must hold true when v is adjacent to both h_{p_1-1} and h_{p_2+1}. In the following, $2 \le p_2 - p_1 \le 3$, and we consider $v \not\sim h_{p_1-1}$, while $v \not\sim h_{p_2+1}$ follows by symmetry. If $p_2 - p_1 = 2$, then we return • $H \cup \{v\}$ as a $K_{2,3}$ when $|H| = 4$; • $H \cup \{v\}$ as a twin-C_5 when $|H| = 5$ and $|N_H[v]| = 2$; • $H \cup \{v\}$ as an FIS-1 when $|H| = 5$ and $|N_H[v]| = 3$; or • $\{h_{p_1-2}, h_{p_1-1}\dots, h_{p_2}, v\}$ as a domino when $|H| > 5$. Otherwise, $p_2 - p_1 = 3$, and we return • $H \cup \{v\}$ as a twin-C_5 when $|H| = 5$; • $H \cup \{v\}$ as an FIS-2 when $|H| = 6$ and $v \not\sim h_{p_2+1}$; • $vh_{p_1}h_{p_1-1}h_{p_1-2}$ and h_{p_2-1} as a C^* when $|H| = 6$ and $v \sim h_{p_2+1}$; or • $vh_{p_1}h_{p_1-1}h_{p_1-2}$ and h_{p_2-1} as a C^* when $|H| > 6$.

The test of a vertex v takes $O(d(v))$ time. Therefore, all vertices can be tested in $O(n+m)$ time. For the vertex failing the test, the indices p_1, p_2, p_3, and p_4 can be detected in $O(|H| + d(v))$ time. The rest of the construction takes $O(|H|)$ time. This concludes the proof. □

Now consider a hole H satisfying property P1. For every vertex $v \in G$, we assign canonical indices to the ends of the path induced by $N_H[v]$ as follows.

Definition 1. Let H be a hole satisfying properties P1. For each vertex $v \in G$, we denote by $\texttt{first}(v)$ and $\texttt{last}(v)$ respectively the indices of the counterclockwise and clockwise ends of the path induced by $N_H[v]$ on H satisfying

- $-|H| < \texttt{first}(v) \le 0 \le \texttt{last}(v) < |H|$ if $h_0 \in N_H[v]$; or
- $0 < \texttt{first}(v) \le \texttt{last}(v) < |H|$, otherwise.

It is possible that $\texttt{last}(v) = \texttt{first}(v)$, when $|N_H[v]| = 1$. In general, $\texttt{last}(v) - \texttt{first}(v) = |N_H[v]| - 1$, and $v = h_i$ or $v \sim h_i$ for each i with $\texttt{first}(v) \le i \le \texttt{last}(v)$. The indices $\texttt{first}(v)$ and $\texttt{last}(v)$ can be easily retrieved from Lemma 2.2. With them we can check the adjacency between v and any vertex $h_i \in H$ in constant time. Now consider property P2, which is on the neighbors of more than one vertices on H.

Lemma 2.3. *Given a hole H, we can in $O(n + m)$ time either determine that H satisfies property* P2*, or produce a subgraph of G that is in $\mathcal{F}$.*

Proof. We call first the algorithm of Lemma 2.2. We may assume that H satisfies property P1, and hence we have the $2n$ indices $\texttt{last}(v)$ and $\texttt{first}(v)$ for all vertices. With them we can check for each edge $uv \in E(G)$ whether (i) $N_H[u] \cap N_H[v] \neq \emptyset$, and (ii) if neither of $N_H[u]$ and $N_H[v]$ is a subset of the other, then precisely one of $h_{\texttt{first}(u)}$ and $h_{\texttt{last}(u)}$ is in $N_H[v]$. If all edges pass the test, then H satisfies P2, and we are done. Otherwise, let uv be an edge that failed the test, and we construct a subgraph of G that is in $\mathcal{F}$ as follows.

If $N_H[u] \cap N_H[v] = \emptyset$ (i.e., the edge uv failed the test because of condition (i)), then neither of u and v can be in H. The indices $\texttt{last}(u)$, $\texttt{first}(u)$ and $\texttt{last}(v)$, $\texttt{first}(v)$ partition H into four sub-paths, two of which are induced by $N_H[u]$ and $N_H[v]$. Denote by P_1 and P_2 the other two sub-paths; their ends are adjacent to u and v respectively, while their inner vertices, if any, are adjacent to neither u nor v. See Fig. 4(a).

Assume first that both P_1 and P_2 are of length 1, then $|H| = |N_H[u]| + |N_H[v]|$. If u is adjacent to a single vertex h_i in H, (hence $|N_H[v]| \ge 3$,) then we return $\{h_i, h_{i-1}, h_{i+1}, u, v\}$ as a $K_{2,3}$. A symmetric argument applies when $|N_H[v]| = 1$. If $|N_H[u]| = |N_H[v]| = 2$, then we return $H \cup \{u, v\}$ as a $\overline{C_6}$. It must be in some case above if $|H| = 4$, and henceforth we assume $|H| > 4$. If u is adjacent to only h_i and h_{i+1} in H, (hence $|N_H[v]| \ge 3$,) then we return $\{h_{i-1}, h_i, h_{i+1}, h_{i+2}, u, v\}$ as an FIS-1. A symmetric argument applies when $|N_H[v]| = 2$. Now that both $|N_H[u]|$ and $|N_H[v]|$ are at least 3, we return $\{u, v, h_{\texttt{first}(u)}, h_{\texttt{last}(u)}, h_{\texttt{first}(v)}, h_{\texttt{last}(v)}\}$ as a domino.

Assume now that, without loss of generality, the length of P_2 is at least 2. We can return vuP_1 and $h_{\texttt{last}(v)+1}$ (when both $|N_H[u]| > 1$ and $|N_H[v]| > 1$) or vuP_1 and $h_{\texttt{last}(v)+2}$ (when the length of P_2 is larger than 3) as a C^*. A symmetric argument applies when the length of P_1 is larger than 3. In the remaining cases, we assume without loss of generality, $|N_H[u]| = 1$, and the lengths of both paths P_1 and P_2 are at most 3. Consequently, $|H\backslash N_H[v]| \leq 5$. If the length of P_1 is at least 2, then $|H\backslash N_H[v]| \geq 3$. We return $\{u, v, h_{\texttt{first}(v)-1}, h_{\texttt{last}(v)+1}\} \cup N_H[v]$ as a † when $|N_H[v]| > 1$. Now that $|N_H[v]| = 1$, then $|H| \leq 6$, and we return • $(H\backslash N_H[v]) \cup \{u, v\}$ as a long claw when $|H| = 6$; • $H \cup \{u, v\}$ as a twin-C_5 when $|H| = 4$; or • $H \cup \{u, v\}$ as an FIS-2 when $|H| = 5$. In the final case, P_1 has length 1, which means that neither u nor v is adjacent to $h_{\texttt{first}(u)-1}$. If $v \sim h_{\texttt{first}(u)-2}$, then we return $\{h_{\texttt{first}(u)-2}, h_{\texttt{first}(u)-1}, h_{\texttt{first}(u)}, h_{\texttt{first}(u)+1}, u, v\}$ as • an FIS-1 when $|H| = 4$; or • a twin-C_5 when $|H| > 4$. If $v \not\sim h_{\texttt{first}(u)-2}$, then we return • $H \cup \{u, v\}$ as a domino when $|H| = 4$; or • (uvh_1h_0u) and h_{-2} as a C^* when $|H| > 4$.

In the following the edge uv failed the test because of condition (ii). Then v is adjacent all vertices in $H\backslash N_H[u]$ as well as $h_{\texttt{last}(u)}$ and $h_{\texttt{first}(u)}$, and we can return $uh_{\texttt{last}(v)}h_{\texttt{last}(v)+1}\cdots h_{\texttt{first}(v)}$, which is a hole, and v as a wheel. See Fig. 4(b).

We can check each edge in constant time, and all edges in $O(n+m)$ time. The subsequent construction of the subgraph takes $O(n)$ time. The proof is now complete. □

If both properties are satisfied by H, then the pattern of the four ends of $N_H[u]$ and $N_H[v]$ can be summarized as follows. In the case that they are all distinct, if we traversing H from $h_{\texttt{first}(u)}$ to both directions, we must meet $h_{\texttt{first}(v)}$ in one of them; in other words, we cannot meet both $h_{\texttt{last}(u)}$ and $h_{\texttt{last}(v)}$ before $h_{\texttt{first}(v)}$ (see Fig. 4). It is similar when some of them coincide. Lemmas 2.2 and 2.3 imply that in a normal Helly circular-arc graph, every hole satisfies properties P1 and P2.

Since we have assumed that G is not chordal, we can call the algorithm of Tarjan and Yannakakis [22] to detect a hole H. We then use the algorithms of Lemmas 2.2 and 2.3 to check whether the hole satisfies properties P1 and P2. A subgraph of G that is in $\mathcal{F}$, once detected, will terminate the recognition of the graph. Henceforth we may assume that H satisfies properties P1 and P2. Let $T := N[h_0]$ and $\overline{T} := V(G)\backslash T$. As we have alluded to earlier, we want to duplicate T and append them to "different sides of $\overline{T}$". One may be reminded that $\overline{T}$ should induce an interval graph, though we do not want to test it at this moment. Each edge between $v \in T$ and $u \in \overline{T}$ will be carried by only one copy of T, and this is determined by the direction of this edge, which is specified as follows. Note that u is adjacent to either $\{h_{\texttt{first}(v)}, \ldots, h_{-1}\}$ or $\{h_1, \ldots, h_{\texttt{last}(v)}\}$ but not both (property P2). The edge uv is said to be clockwise from T if u is adjacent to any h_i with $1 \leq i \leq \texttt{last}(v)$, and counterclockwise otherwise. Let E_c (resp., E_{cc}) denote the set of clockwise (resp., counterclockwise) edges from T, and let T_c (resp., T_{cc}) denote the subsets of vertices of T that are incident to edges in E_c (resp., E_{cc}).

For example, $\{h_{-1}, h_0, h_1\} \subseteq T$, where h_{-1} and h_1 are in T_{cc} and T_c respectively, witnessed by $h_{-2}h_{-1} \in E_{cc}$ and $h_1h_2 \in E_c$. Note that h_{-2} and h_2 refer to the same vertex if H is a C_4; hence, a vertex in $\overline{T}$ can be incident to edges both in E_{cc} and E_c. Likewise, a vertex $v \in T$ may belong to both T_{cc} and T_c; such a vertex must be adjacent to both h_{-1} and h_1. However, an edge cannot be in both E_{cc} and E_c, i.e., $\{E_{cc}, E_c\}$ partitions edges between T and $\overline{T}$.

We have now all the details for the definition of the auxiliary graph $\mho(G)$.

Definition 2. Let H be a hole satisfying properties P1 and P2 and let $T := N[h_0]$. The vertex set of $\mho(G)$ consists of $\overline{T} \cup L \cup R \cup \{w\}$, where L and R are distinct copies of T, i.e., for each $v \in T$, there are a vertex v^l in L and another vertex v^r in R, and w is a new vertex distinct from $V(G)$. The edge set of $\mho(G)$ consists of wv^l for every $v \in T_{cc}$, and for each edge $uv \in E(G)$

- an edge uv if neither u nor v is in T;
- two edges u^lv^l and u^rv^r if both u and v are in T; or
- an edge uv^l or uv^r if $uv \in E_c$ or $uv \in E_{cc}$ respectively ($v \in T$ and $u \in \overline{T}$).

Lemma 2.4. *The numbers of vertices and edges of $\mho(G)$ are upper bounded by $2n$ and $2m$ respectively. Given a hole H satisfying properties P1 and P2, an adjacency list representation of $\mho(G)$ can be constructed in $O(n+m)$ time. In the same time, vertex sets T_c, T_{cc} and edge sets E_c, E_{cc} can be produced.*

Proof. The vertices of the auxiliary graph $\mho(G)$ include $\overline{T}$, two copies of T, and w. So the number of vertices is $2|T|+|\overline{T}|+1 = |V(G)|+|T|+1 \leq 2n$. In $\mho(G)$, there are two edges derived from every edge of $G[T]$ and one edge from every other edge of G. All other edges are incident to w, and there are T_{cc} of them. Therefore, the number of edges is $|E(G)|+|E(G[T])|+|T_{cc}| \leq |E(G)|+|E(G[T])|+|E_{cc}| < 2m$.

For the construction of $\mho(G)$, we use the procedure described in Fig. 5. Step 1 adds vertex sets L and R (step 1.1) as well as those edges induced by them (step 1.2.1), and finds the open neighborhood of T (step 1.2.2): after all vertices in T have been scanned in step 1, $N(T) = NT$. Step 2 scans edges between T and $N(T)$, and adds them to E_{cc} or E_c accordingly; meanwhile, it also detects T_{cc} and T_c. Since $u \not\sim h_0$ and since H satisfies P2, $\texttt{first}(v)$ and $\texttt{last}(v)$ cannot be both 0, and u is adjacent to some vertex in $\{h_{\texttt{first}(v)}, \ldots, h_{-1}\}$ and $\{h_1, \ldots, h_{\texttt{last}(v)}\}$. On the other hand, by P1 and P2, u is adjacent to only one of the two sets. In the first case, $|H|+\texttt{first}(v) \leq \texttt{last}(u)$, while in the second case, $0 < \texttt{first}(u) \leq \texttt{last}(v)$. They are handled by steps 2.1.1 and 2.1.2 respectively, which put the edge uv is in E_{cc} and E_c, and the vertex v is in T_{cc} and T_c accordingly. Also in step 2, vertices of T are removed from the adjacency lists of $N(T)$. Steps 3 and 4 add vertex w and edges incident to it. Step 5 cleans T and finishes the algorithm. The dominating steps are 1 and 2, each of which checks every edge at most twice, and hence the total running time is $O(n+m)$. This concludes the proof of the lemma. □

Note that H is mapped into an induced path $wh^l_{-1}h^l_0h^l_1h_2\cdots h_{-2}h^r_{-1}h^r_0h^r_1$, denoted by P_H. Because of properties P1 and P2, the hole H can be viewed as the axis for vertices of G. Likewise, P_H can be viewed as the axis for vertices of $\mho(G)$. The following are immediate from properties P1 and P2.

INPUT: a graph G and a hole H satisfying properties P1 and P2.
OUTPUT: the auxiliary graph $\mho(G)$.

0. $T \leftarrow \emptyset$; $NT \leftarrow \emptyset$; $T_c \leftarrow \emptyset$; $T_{cc} \leftarrow \emptyset$; $E_c \leftarrow \emptyset$; $E_{cc} \leftarrow \emptyset$;
   for each $v \in V(G)$ **do** compute $\texttt{first}(v)$ and $\texttt{last}(v)$;
   for each $v \in N[h_0]$ **do** mark v "in T";
1. **for each** $v \in N[h_0]$ **do**
1.1. add vertices v^l and v^r;
1.2. **for each** $u \in N(v)$ **do**
1.2.1. **if** u is marked "in T" **then** add u^l to $N(v^l)$ and u^r to $N(v^r)$;
1.2.2. **else if** u is not marked "in NT" **then** mark u "in NT" and $NT \leftarrow NT \cup \{u\}$;
2. **for each** $u \in NT$ **do**
2.1. **for each** $v \in N(u)$ that is marked "in T" **do**
   \\ $\texttt{first}(v) \leqslant 0 \leqslant \texttt{last}(v)$, *but the equalities cannot be both true (property P2).*
2.1.1. **if** $|H| + \texttt{first}(v) \leqslant \texttt{last}(u)$ **then**
   $E_{cc} \leftarrow E_{cc} \cup \{uv\}$;
   replace v by v^l in $N(u)$ and add u to $N(v^l)$;
   if v is not marked "in T_{cc}" **then** mark v and $T_{cc} \leftarrow T_{cc} \cup \{v\}$;
2.1.2. **else** \\ $0 < \texttt{first}(u) \leqslant \texttt{last}(v)$.
   $E_c \leftarrow E_c \cup \{uv\}$;
   replace v by v^r in $N(u)$ and add u to $N(v^r)$;
   if v is not marked "in T_c" **then** mark v and $T_c \leftarrow T_c \cup \{v\}$;
3. add a new vertex w;
4. **for each** $v \in T_{cc}$ **do** add w to $N(v^l)$ and add v^l to $N(w)$;
5. remove T and **return** the constructed graph as $\mho(G)$.

Fig. 5. Procedure for constructing $\mho(G)$ (Lemma 2.4).

Proposition 2.5. *If H satisfies properties* P1 *and* P2, *then*

(I) *for every vertex v of $\mho(G)$, the neighbors of v on P_H induce a proper nonempty sub-path of P_H; and*
(II) *for any pair of adjacent vertices u, v of $\mho(G)$, either one of $N_{P_H}[u]$ and $N_{P_H}[v]$ is a subset of the other, or precisely one end of $N_{P_H}[u]$ is contained in $N_{P_H}[v]$.*

In an interval model, each vertex v corresponds to a closed interval $I(v) = [\texttt{lp}(v), \texttt{rp}(v)]$, where $\texttt{lp}(v) < \texttt{rp}(v)$ are the *left endpoint* and *right endpoint* of $I(v)$ respectively. In a circular-arc model with perimeter ℓ, each vertex v corresponds to a closed arc $A(v) = [\texttt{ccp}(v), \texttt{cp}(v)]$, where $0 \le \texttt{ccp}(v), \texttt{cp}(v) < \ell$ and $\texttt{ccp}(v) \neq \texttt{cp}(v)$. We say that $\texttt{ccp}(v)$ and $\texttt{cp}(v)$ are *counterclockwise endpoint* and *clockwise endpoint* of $A(v)$ respectively. It is worth noting that possibly $\texttt{cp}(v) < \texttt{ccp}(v)$; such an arc necessarily contains the point 0. In the rest of this section, we assume that a circular-arc model is always given in the same direction with H, i.e., $\texttt{ccp}(h_1)$ is contained in $A(h_0)$. In general, for any vertex v, the endpoints $\texttt{ccp}(v)$ and $\texttt{cp}(v)$ satisfy

$$\texttt{ccp}(v) \in \left(\texttt{cp}(h_{\texttt{first}(v)-1}), \texttt{cp}(h_{\texttt{first}(v)})\right] \subset A(h_{\texttt{first}(v)}) \tag{1}$$

$$\text{and} \quad \texttt{cp}(v) \in \left[\texttt{ccp}(h_{\texttt{last}(v)}), \texttt{ccp}(h_{\texttt{last}(v)+1})\right) \subset A(h_{\texttt{last}(v)}). \tag{2}$$

For example, see the arc for v_2 in Fig. 6(a), where $\texttt{first}(v_2) = -1$ and $\texttt{last}(v_2) = 2$.

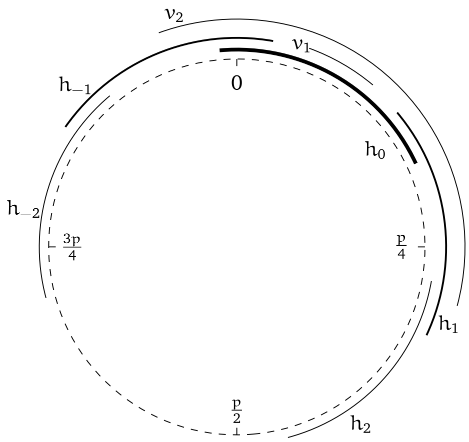
The definition of $\mho(G)$ is motivated by the following lemma. Although it is already implied by Theorem 1.3, a constructive proof is presented here in the hope that it may help elucidate $\mho(G)$.

Lemma 2.6. *Let $\mho(G)$ be the auxiliary graph defined with a hole H satisfying properties* P1 *and* P2. *If G is a normal Helly circular-arc graph, then $\mho(G)$ is an interval graph.*

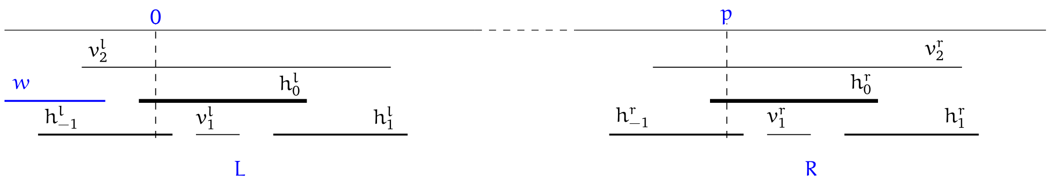
Proof. Let $\mathcal{A}$ be a normal and Helly circular-arc model for G, and let p be the perimeter of the circle in the model. The union of arcs for T, i.e., $\bigcup_{v \in T} A(v)$, does not cover the circle. We may assume $0 \in A(h_0)$ and there is no endpoint of any arc lying in $(\texttt{ccp}(h_0), 0]$; this can be achieved by rotating all arcs in $\mathcal{A}$. As a consequence, $0 \in A(v)$ for every $v \in T_{cc}$. An interval model for $\mho(G)$ can be obtained from $\mathcal{A}$ by setting

- $I(v^l) := [\texttt{ccp}(v) - p, \texttt{cp}(v)]$ and $I(v^r) := [\texttt{ccp}(v), \texttt{cp}(v) + p]$ for $v \in T$ and $0 \in A(v)$;
- $I(v^l) := [\texttt{ccp}(v), \texttt{cp}(v)]$ and $I(v^r) := [\texttt{ccp}(v) + p, \texttt{cp}(v) + p]$ for $v \in T$ but $0 \notin A(v)$;
- $I(u) := [\texttt{ccp}(u), \texttt{cp}(u)]$ for $u \in \overline{T}$; and
- $I(w) := [-p, \max_{x \in \overline{T}} \texttt{cp}(x) - p]$.

It is easy to use definition to verify that these intervals represent $\mho(G)$. □



(a) Part of a normal and Helly circular-arc model for G.



(b) An interval model for $\mho(G)$ derived from (a).

Fig. 6. Illustration for Lemma 2.6.

The main thrust of our recognition algorithm will be a process that retrieves a circular-arc model for $\mho(G)$ from an interval model for G. This is nevertheless far more involved than its opposite direction given in Lemma 2.6. For example, in an interval model provided by known recognition algorithms for interval graphs, arcs $I(v^l)$ and $I(v^r)$ may not have the same length, and even they do, they may have different offsets from $\mathtt{lp}(v_0^l)$ and $\mathtt{lp}(v_0^r)$. Even though the construction given in the proof of Lemma 2.6 ensures that such an interval model always exists, it is not clear how to obtain it efficiently.

Theorem 2.7. *Let $\mho(G)$ be the auxiliary graph defined with a hole H satisfying properties* P1 *and* P2*. If $\mho(G)$ is an interval graph, then we can in $O(n+m)$ time build a circular-arc model for G. Moreover, the model is normal.*

Proof. By Lemma 2.4, $\mho(G)$ has $O(n)$ vertices and $O(m)$ edges. Thus, we can in $O(n+m)$ time build an interval model $\mathcal{I}$ for $\mho(G)$. Let $0 = \mathtt{rp}(w)$ and $a = \max_{u \in \overline{T}} \mathtt{rp}(u)$. Without loss of generality, assume the path P_H goes "from left to right" in $\mathcal{I}$. Then $a > 0$. We use $\mathcal{I}$ to construct a set of arcs for $V(G)$ on a circle of perimeter $a + \epsilon$ (where ϵ is a small positive number such that no endpoint of $\mathcal{I}$ lies in $(a, a+\epsilon]$) as follows:

$$A(v) := \begin{cases} [\mathtt{lp}(v^r), \mathtt{rp}(v^l)] & \text{if } v \in T_{cc}, \\ I(v^l) & \text{if } v \in T \backslash T_{cc}, \\ I(v) & v \in \overline{T}. \end{cases} \tag{3}$$

By property P1 and the definition of $\overline{T}$, every $u \in \overline{T}$ is adjacent to $H \backslash \{h_0\}$; therefore, $\mathtt{rp}(h_0^l) < \mathtt{lp}(u) < \mathtt{rp}(u) \le a$. Note that $v^l \sim w$ and $v^r \sim \overline{T}$ for every $v \in T_{cc}$; as a result, $\mathtt{lp}(v^l) < 0$ if and only if $\mathtt{lp}(v^r) < a$ if and only if $v \in T_{cc}$.

It remains to verify that the arcs given by (3) represent G, i.e., a pair of vertices u, v of G is adjacent if and only if $A(u)$ and $A(v)$ intersect. This holds trivially when neither u nor v is in T, then $A(u) \cap A(v) = I(u) \cap I(v)$. Hence, we may assume without loss of generality that $v \in T$. Consider first that u is also in T, then $u \sim v$ in G if and only if $u^l \sim v^l$ in $\mho(G)$.

- If both $u, v \in T_{cc}$, then both $I(u^l)$ and $I(v^l)$ contain 0, hence $u^l \sim v^l$ in $\mho(G)$; on the other hand, noting $\mathtt{rp}(v^l) < \mathtt{lp}(v^r)$ and $\mathtt{rp}(u^l) < \mathtt{lp}(u^r)$, both $A(u)$ and $A(v)$ contains the point 0 and thus intersect.
- If neither u nor v is in T_{cc}, then $A(u) \cap A(v) = I(u^l) \cap I(v^l)$, which is nonempty if and only if $u \sim v$ in G.
- Otherwise, precisely one of u and v is in T_{cc}; without loss of generality, let it be v. Then $\mathtt{lp}(v^l) < 0 < \mathtt{lp}(u^l)$, and $u \sim v$ in G if and only if $\mathtt{lp}(u^l) < \mathtt{rp}(v^l)$, which is equivalent to $A(u) \cap A(v) = [\mathtt{lp}(u^l), \mathtt{rp}(v^l)] \neq \emptyset$.

Consider now that u is not in T, and then $u \sim v$ in G if and only if either $u \sim v^l$ or $u \sim v^r$ in $\mho(G)$. In the case $u \sim v^l$, we have $\mathtt{lp}(v^l) < \mathtt{rp}(h_0^l) < \mathtt{lp}(u) < \mathtt{rp}(v^l)$, and then $A(u) \cap A(v) = [\mathtt{lp}(u), \mathtt{rp}(v^l)] \neq \emptyset$. In the case $u \sim v^r$, we have $\mathtt{lp}(v^r) < \mathtt{rp}(u) \le a$, and then $A(u) \cap A(v) = [\mathtt{lp}(v^r), \mathtt{rp}(u)] \neq \emptyset$. On the other hand, if $u \not\sim v$ in G, then $\mathtt{lp}(v^l) < \mathtt{rp}(v^l) < \mathtt{lp}(u) < \mathtt{rp}(u) < \mathtt{lp}(v^r) < \mathtt{rp}(v^r)$, hence $A(u) \cap A(v) = \emptyset$.

INPUT: a graph G.
OUTPUT: either a normal and Helly circular-arc model for G, or a subgraph of G that is in $\mathcal{F}$.

1. **if** G is chordal **then**
   return either an interval model for G or a subgraph in Fig. 1;
2. find a hole H;
3. **if** H does not satisfy any of P1 and P2 **then return** a subgraph of G that is in $\mathcal{F}$;
4. construct the auxiliary grpah $\mho(G)$; \\ *Lemma 2.4.*
5. **if** $\mho(G)$ is not an interval graph **then return** a subgraph of G that is in $\mathcal{F}$; \\ *Theorem 1.3.*
6. build a normal circular-arc model $\mathcal{A}$ for G; \\ *Theorem 2.7.*
7. **if** $\mathcal{A}$ is Helly **then return** $\mathcal{A}$;
   else return the wheel found by Lemma 2.8.

Fig. 7. The recognition algorithm for normal Helly circular-arc graphs.

During the construction, each arc is scanned once; hence, the running time is $O(n)$. We now argue that $\mathcal{A}$ is normal by contradiction. Suppose that two arcs $A(u)$ and $A(v)$ in $\mathcal{A}$ cover the circle, then by (1) and (2), u is necessarily adjacent to both $h_{\mathtt{first}(v)}$ and $h_{\mathtt{last}(v)}$. But this contradicts the fact that H satisfies property P2. □

However, the circular-arc model $\mathcal{A}$ constructed in Theorem 2.7 may not be Helly. It can be decided in linear time whether $\mathcal{A}$ is Helly, and if not, the algorithm of [15] can find in the same time three arcs that cover the circle. This case is handled by the following lemma: note that any three arcs covering the model represent such a set U of vertices.

Lemma 2.8. *Given a set U of three pairwise adjacent vertices such that $H \subseteq N(U)$ and for each $v \in U$, both $h_{\mathtt{last}(v)}$ and $h_{\mathtt{first}(v)}$ are adjacent to at least two vertices of U, we can in $O(n+m)$ time find a wheel that is a subgraph of G.*

Proof. Let $U = \{x, y, z\}$. Since H satisfies P2, none of $N_H[x]$, $N_H[y]$, and $N_H[z]$ contains another as a subset: suppose, e.g., $N_H[x] \subseteq N_H[y]$, then both ends of $N_H[z]$ would be in $N_H[y]$. Without loss of generality, assume that $h_{\mathtt{last}(x)} \in N_H[y]$, which implies $h_{\mathtt{first}(y)} \in N_H[x]$. Then both $h_{\mathtt{first}(x)}$ and $h_{\mathtt{last}(y)}$ are adjacent to z. The $h_{\mathtt{last}(y)}$-$h_{\mathtt{first}(x)}$ path in H whose inner vertices are adjacent to neither x nor y makes a hole with x, y. By assumption, z is adjacent to every vertex in the hole, and thus we return this hole and z as a wheel. □

We are now ready to present the recognition algorithm in Fig. 7, and use it to prove Theorem 1.2.

Proof of Theorem 1.2. We use the algorithm described in Fig. 7. Steps 1 and 2 use the algorithms of [22] and [14]. Step 3 uses Lemmas 2.2 and 2.3. Steps 4–6 follow from Lemma 2.4, Theorem 1.3, and Theorem 2.7, respectively. Step 7 uses the algorithm of [15] to verify whether the model $\mathcal{A}$ built in step 6 is Helly; if not, then it can detect three arcs whose union covers the circle and calls Lemma 2.8. All these steps can be done in $O(n+m)$ time. □

The major step of this recognition algorithm is to build an interval model for $\mho(G)$, thereby considerably simpler than the known algorithm [15]. It is worth noting that if we are after a recognition algorithm (with positive certificate only), then we can simply return "NO" instead of a subgraph of G that is in $\mathcal{F}$ in the algorithm. In particular, we do not need Theorem 1.3, and the correctness of returning "NO" in step 5 is ensured by Lemma 2.6. Therefore, the algorithm is already complete.

3. Proof of Theorem 1.3

Recall that the auxiliary graph $\mho(G)$ is constructed with a hole H satisfying properties P1 and P2. In principle, any hole satisfying them can be used, and any vertex in the hole can be the h_0. But for the convenience of presentation of this section, we require the special vertex h_0 of the hole to satisfy some additional condition. If some vertex v is adjacent to four or more vertices in H, i.e., $\mathtt{last}(v) - \mathtt{first}(v) > 2$, then $v \notin H$, and we can use v to produce a strictly shorter hole, bypassing the inner vertices of $N_H[v]$. The condition that h_0 cannot be bypassed as such is formalized as our third property:

(P3) For every vertex v, $\{h_{-1}, h_0, h_1\} \subseteq N_H[v]$ if and only if $N_H[v] = \{h_{-1}, h_0, h_1\}$.

We point out that unlike properties P1 and P2 that are satisfied by all holes in a normal Helly circular-arc graph, P3 is only satisfied by some holes, e.g., those with the smallest length in the graph. We need first a linear-time procedure for finding such a hole.

Lemma 3.1. *Given a hole H of a graph G, we can in $O(n+m)$ time find either a hole H satisfying properties* P1–P3*, or a subgraph of G that is in $\mathcal{F}$.*

INPUT: a graph G and a hole H of G.
OUTPUT: a hole satisfying P1-P3 or a subgraph of G that is in $\mathcal{F}$.

1. call Lemma 2.2 to compute $\texttt{first}(v)$ and $\texttt{last}(v)$ for all vertices (w.r.t. H);
   if a subgraph of G that is in $\mathcal{F}$ is found **then return** it;
2. $h \leftarrow h_0$; $a \leftarrow -1$; $b \leftarrow 1$;
3. **for each** $v \in V(G) \setminus H$ **do**
3.1. **if** $(\texttt{first}(v) < a$ and $\texttt{last}(v) \geqslant b)$ or $(\texttt{first}(v) = a$ and $\texttt{last}(v) > b)$ **then**
   $h \leftarrow v$; $a \leftarrow \texttt{first}(v)$; $b \leftarrow \texttt{last}(v)$; \\ $\texttt{first}(v) < 0 < \texttt{last}(v)$.
4. call Lemmas 2.2 and 2.3 with the hole $hh_bh_{b+1}\cdots h_a$;
   if a subgraph of G that is in $\mathcal{F}$ is found **then return** it;
   else return $hh_bh_{b+1}\cdots h_a$, where h is the new h_0.

Fig. 8. Procedure for finding a hole satisfying properties P1–P3 (Lemma 3.1).

INPUT: a subgraph $F \in \mathcal{F}$ of $\mho(G)$.
OUTPUT: a subgraph of G that is in $\mathcal{F}$.

5.0. **if** T_{cc} or T_c is not a clique **then** call Lemma 3.6 and **return** the found subgraph;
5.1. **if** F is a hole **then** call Lemma 3.7 and **return** the found subgraph;
5.2. **else** call Lemma 3.8 and **return** the found subgraph.

Fig. 9. Procedure for finding a subgraph of G that is in $\mathcal{F}$ (step 5 of Fig. 7).

Proof. We apply the procedure described in Fig. 8. Step 3 greedily searches for an inclusion-wise maximal $N_H[v]$ satisfying $\texttt{first}(v) \leq -1$ and $1 \leq \texttt{last}(v)$. Initially, $a = \texttt{first}(h_0) = -1$ and $b = \texttt{last}(h_0) = 1$. Each iteration of step 3 checks an unexplored vertex v in $V(G)\backslash H$. If either condition of step 3.1 is satisfied, then $N_H[v]$ properly contains $\{h_a, h_{a+1}, \ldots, h_b\}$, and a and b are updated to be $\texttt{first}(v)$ and $\texttt{last}(v)$ respectively. Note that the values of a and b are non-increasing and nondecreasing respectively. Thus, no previously explored vertex is adjacent to all of $\{h_a, h_{a+1}, \ldots, h_b\}$. After step 3, all vertices have been explored, and the hole $hh_bh_{b+1}\cdots h_a$ satisfies P3. Step 4 then calls Lemmas 2.2 and 2.3 to check whether this new hole satisfies properties P1 and P2 as well. If not, then they return a subgraph of G that is in $\mathcal{F}$.

Steps 1 and 4 take $O(n+m)$ time, while step 3 takes $O(n)$ time (each vertex can be checked in $O(1)$ time). It follows that the running time of the procedure is $O(n+m)$. □

This linear-time procedure can be called in place of step 3 of Fig. 7, and it does not impact the asymptotic time complexity of the algorithm, which remains linear. Recall that Theorem 1.3 is only called in step 5 of our recognition algorithm (Fig. 7). In the rest of this section, we prove Theorem 1.3 assuming that H satisfies properties P1–P3, for which we apply the procedure outlined in Fig. 9. In particular, it first checks whether T_{cc} and T_c both induce cliques. If either of them does not, then a pair of nonadjacent vertices in it can be found in linear time, and the procedure calls Lemma 3.6.[1] After that, it proceeds to call Lemma 3.7 or 3.8 based on whether $\mho(G)$ is chordal or not. Recall that if $\mho(G)$ is not chordal, then the algorithm of Tarjan and Yannakakis [22] can find a hole; and if it is chordal but not an interval graph, then the algorithm of Lindzey and McConnell [14] can find a subgraph in Fig. 1, both running in linear time.

To prove Lemmas 3.6–3.8, we need some notation and simple facts. Each vertex x of $\mho(G)$ different from w is uniquely defined by a vertex of G. This vertex is denoted by $\phi(x)$, and we say that x is *derived from* $\phi(x)$. For example, $\phi(v^l) = \phi(v^r) = v$ for $v \in T$. By abuse of notation, we will use the same letter for a vertex $u \in \overline{T}$ of G and the unique vertex of $\mho(G)$ derived from u; its meaning is always clear from the context. Therefore, $\phi(u) = u$ for $u \in \overline{T}$, and in particular, $\phi(h_i) = h_i$ for $i = 2, \ldots, |H|-2$. We can mark $\phi(x)$ for each vertex of $\mho(G)$ during its construction. The function ϕ is generalized to a set U of vertices that does not contain w, i.e., $\phi(U) = \{\phi(v) : v \in U\}$. We point out that possibly $|\phi(U)| \neq |U|$.

By the construction of $\mho(G)$, if a pair of vertices x and y (different from w) is adjacent in $\mho(G)$, then $\phi(x)$ and $\phi(y)$ must be adjacent in G as well. The converse is not necessarily true, e.g., for any vertex $v \in T_c$ and edge $uv \in E_c$, we have $u \not\sim v^r$, and for any pair of adjacent vertices $u, v \in T$, we have $u^l \not\sim v^r$ and $u^r \not\sim v^l$. We say that a pair of vertices x, y of $\mho(G)$ is a *broken pair* if $\phi(x) \sim \phi(y)$ in G but $x \not\sim y$ in $\mho(G)$. By definition, w does not participate in any broken pair, and at least one vertex of a broken pair is in $L \cup R$. Note that if a connected subgraph F of $\mho(G)$ contains both v^l and v^r for some $v \in T$ (i.e., $|F| > |\phi(F)|$), then if must contain two broken pairs.

Proposition 3.2. *Let F be a connected subgraph of $\mho(G)$ that does not contain w or both v^l, v^r for any $v \in T$. Then F is a (not necessarily induced) subgraph of $G[\phi(F)]$, and they are isomorphic if and only if F contains no broken pairs.*

[1] Lemma 3.6 could also be called earlier, e.g., within the construction of $\mho(G)$ (Fig. 5) and immediately following step 2 where T_{cc} and T_c have been found.

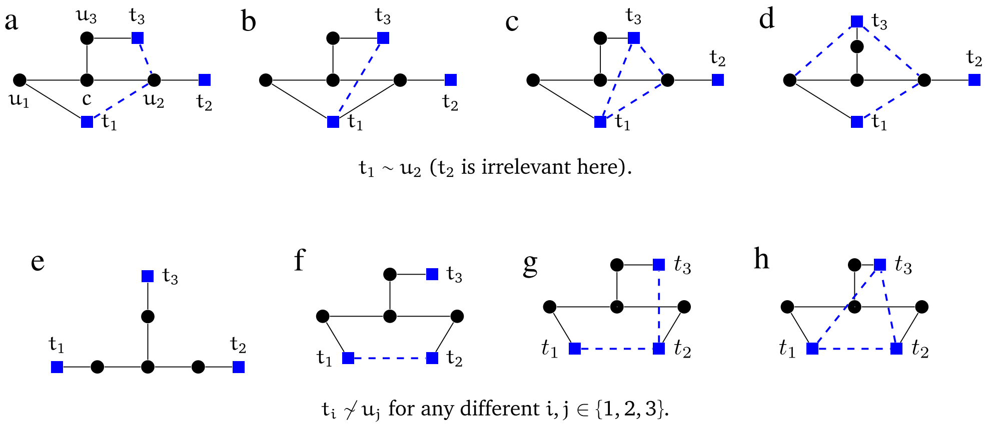


Fig. 10. Illustrations for Proposition 3.5(I).

Proof. If F is free of broken pairs, then the mapping ϕ also defines an isomorphism between F and $G[\phi(F)]$. On the other hand, if F contains a broken pair, then F has strictly less edges than $G[\phi(F)]$, and thus they cannot be isomorphic. □

By the definition of $\mho(G)$, there is no edge between L and R; and for any $v \in T$, there is no vertex adjacent to both v^l and v^r. In other words, for every $v \in T$, the distance between v^l and v^r is at least 3. By Proposition 2.5, every vertex in $h_0^l, h_1^l \dots h_{-1}^r h_0^r$ is adjacent to any v^l-v^r path of $\mho(G)$ for any vertex $v \in T$. Therefore, the following is a direct consequence of Lemma 2.8.

Corollary 3.3. *Let $\mho(G)$ be the auxiliary graph defined with a hole H satisfying properties* P1 *and* P2*, and let $v \in T$. Given a v^l-v^r path of length three, we can in $O(n+m)$ time find a wheel as a subgraph of G.*

Noting that any induced path of length d between a broken pair x, y with $x = v^l$ or v^r can be extended to a v^l-v^r path with length $d + 1$, Proposition 3.2 and Corollary 3.3 have the following consequence.

Corollary 3.4. *Let $F \in \mathcal{F}$ be a subgraph of $\mho(G)$. If the diameter of F is 2, then we can in $O(n+m)$ time find a subgraph of G that is in $\mathcal{F}$.*

Recall that a net is a † on six vertices. The following proposition will be used in both Lemmas 3.6 and 3.8.

Proposition 3.5. (I) *Let $\{c, u_1, u_2, u_3\}$ induce a claw where c has degree three. If there are other (not necessarily distinct) vertices t_1, t_2, and t_3 such that for each $i \in \{1, 2, 3\}$, the vertex t_i is adjacent to u_i but not c, then there is a subgraph in $\mathcal{F}$.*

(II) *Let $\{u_1, u_2, u_3\}$ induce a triangle. If there are three distinct vertices t_1, t_2, and t_3 such that for each $i \in \{1, 2, 3\}$, the vertex t_i is only adjacent to u_i in $\{u_1, u_2, u_3\}$, then there is a subgraph in $\mathcal{F}$.*

Proof. (I) Note that some or all of t_1, t_2, and t_3 may coincide or be adjacent, and for each pair of distinct $i, j \in \{1, 2, 3\}$, vertices u_i and t_j may or may not be adjacent. If $t_1 \sim u_2$, then $t_1u_1cu_2$ is a C_4 (by assumption, $t_1 \not\sim c$ and $u_1 \not\sim u_2$). Based on the adjacency between t_3 and this hole (in particular, u_1, u_2, and t_1), we are in one of the following cases; see the first row of Fig. 10. Let X denote $\{c, u_1, u_2, u_3, t_1, t_3\}$ (here t_2 is irrelevant).

- If t_3 is nonadjacent to the hole $t_1u_1cu_2$, then they make a C^*.
- If t_3 is adjacent to only u_1 or only u_2, then X induces a domino (Fig. 10(a)).
- If t_3 is adjacent to only t_1, then X induces a twin-C_5 (Fig. 10(b)).
- If t_3 is adjacent to t_1 and exactly one of u_1, u_2, then X induces an FIS-1 (Fig. 10(c)).
- If t_3 is adjacent to both u_1 and u_2, then $\{u_1, u_2, u_3, c, t_3\}$ induces a $K_{2,3}$ (Fig. 10(d)). Note that this includes the case $t_1 = t_3$, and the adjacency between t_1 and t_3 is irrelevant otherwise.

The situation is symmetrical if there is other pair of distinct $i, j \in \{1, 2, 3\}$ such that $u_i \sim t_j$. Moreover, one of them must hold true if any two of t_1, t_2, and t_3 coincide, e.g., $t_1 = t_2$ implies $t_1 \sim u_2$. In the remaining cases, neither of them holds true, and we consider the number of edges among the three distinct vertices t_1, t_2, and t_3. See the second row of Fig. 10.

- If t_1, t_2, and t_3 are pairwise nonadjacent, then $\{c, u_1, u_2, u_3, t_1, t_2, t_3\}$ induces a long claw (Fig. 10(e)).
- If there is one edge among t_1, t_2, and t_3, then there is a C_5^*, e.g., $t_1u_1cu_2t_2$ and t_3 when the edge is t_1t_2 (Fig. 10(f)).
- If there are two edges among t_1, t_2, and t_3, then $\{c, u_1, u_2, u_3, t_1, t_2, t_3\}$ induces an FIS-2 (Fig. 10(g)).
- If t_1, t_2, and t_3 are pairwise adjacent, then $\{u_1, u_2, u_3, t_1, t_2, t_3\}$ induces a net (Fig. 10(h)).

(II) We consider the number of edges among the three distinct vertices t_1, t_2, and t_3. See Fig. 11.

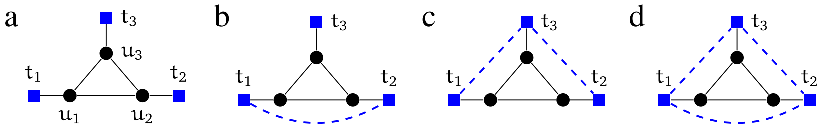


Fig. 11. Illustrations for Proposition 3.5(II).

- If t_1, t_2, and t_3 are pairwise nonadjacent, then $\{u_1, u_2, u_3, t_1, t_2, t_3\}$ induces a net (Fig. 11(a)).
- If there is one edge among t_1, t_2, and t_3, then there is a C_4^*, e.g., $t_1u_1u_2t_2$ and t_3 when the edge is t_1t_2 (Fig. 11(b)).
- If there are two edges among t_1, t_2, and t_3, then $\{u_1, u_2, u_3, t_1, t_2, t_3\}$ induces an FIS-1 (Fig. 11(c)).
- If t_1, t_2, and t_3 are pairwise adjacent, then $\{u_1, u_2, u_3, t_1, t_2, t_3\}$ induces a $\overline{C_6}$ (Fig. 11(d)).

The proof is now complete. □

If G is a normal Helly circular-arc graph, then in a circular-arc model for G, all arcs for T_{cc} contain $\texttt{ccp}(h_0)$. Thus, T_{cc} induces a clique; likewise, T_c also induces a clique. This observation is complemented by the following lemma.

Lemma 3.6. *Given a pair of nonadjacent vertices $u, v \in T_{cc}$ (or T_c), we can in $O(n + m)$ time find a subgraph of G that is in $\mathcal{F}$.*

Proof. We prove the statement on T_{cc}, while a symmetrical argument applies to T_c. By definition, we can find edges $ux, vy \in E_{cc}$, where $x, y \in \overline{T}$. We have then three induced paths $h_0h_1h_2$, h_0ux, and h_0vy. In the following only vertices in these paths concern us; note that some or all of x, y, and h_2 might coincide. By the definition of T_{cc}, both u and v are adjacent to h_{-1}. If they are adjacent to h_1 as well, then we return the hole $uh_{-1}vh_1$ and h_0 as a wheel. Hence, we may assume without loss of generality, $v \not\sim h_1$, and consider whether $u \sim h_1$. If $u \not\sim h_1$, then u, v, and h_1 are pairwise nonadjacent. Noting that h_0 cannot be adjacent to any of x, y, and h_2, we can call Proposition 3.5(I). If $u \sim h_1$, then h_0, h_1, and u make a triangle. By property P3, $N_H[u] = \{h_{-1}, h_0, h_1\}$, and hence $x \neq h_2$. Since $v \in T$ while x and h_2 are not, they are distinct. Noting that (i) v is adjacent to neither u nor h_1; (ii) h_2 is adjacent to neither h_0 nor u; and (iii) x is adjacent to neither h_0 nor h_1 (by the definition of E_{cc}, $x \sim h_{-1}$, and then by property P2, $x \not\sim h_1$), we can call Proposition 3.5(II). Edges ux and vy can be found in $O(n)$ time, and at most 7 vertices are checked subsequently; it thus takes $O(n + m)$ time in total. □

We may assume hereafter that T_{cc} and T_c induce cliques. Let $L_c = \{v^l : v \in T_c\}$ and $R_{cc} = \{v^r : v \in T_{cc}\}$, which are vertices of L and R respectively that are adjacencies to $\overline{T}$ in $\mho(G)$; both induce cliques. Recall that a vertex v is *simplicial* if $N[v]$ induces a clique. Since $N(w)$ is nothing but the vertices in L derived from T_{cc}, it must be a clique of $\mho(G)$. Therefore, w is simplicial and participates in no holes.

Lemma 3.7. *Given a hole C of $\mho(G)$, we can in $O(n + m)$ time find a subgraph of G that is in $\mathcal{F}$.*

Proof. Let us first take care of some trivial cases. If C is contained in $\overline{T}$ or L or R, then by the construction of $\mho(G)$, the set $\phi(C)$ of vertices induces a hole of G. This hole is either nonadjacent (when $C \subseteq \overline{T}$) or completely adjacent (when $C \subseteq L$ or $C \subseteq R$) to h_0 in G, whereupon we can return $\phi(C)$ and h_0 as a C^* or wheel respectively. Since L and R are nonadjacent, one of the cases above must hold true if C is disjoint from $\overline{T}$. Henceforth we may assume that C intersects $\overline{T}$ and, without loss of generality, L—noting that w is neither in the hole C nor used in the following argument, hence a symmetrical argument applies when C intersects R.

Since L_c is a clique, and since no vertex in $L \setminus L_c$ is adjacent to $\overline{T} \cup R$, there are at most two vertices in $C \cap L$, which have to be in L_c. Let u^l be a vertex in $C \cap L$, and let $a := \texttt{last}(u)$. Since H satisfies property P2, u cannot be adjacent to h_{-1}, which means $a \leq |H| - 2$. Let us use $h'_0, h'_1, h'_2, \ldots, h'_{|H|-2}, h'_{|H|-1}$ as aliases for $h^l_0, h^l_1, h_2, \ldots, h_{|H|-2}, h^r_{|H|-1}$ in the rest of the proof.

The first case we consider is $h'_a \in C$. Note that a cannot be 1, as otherwise by property P2, the other neighbor of u^l in C is adjacent to h'_1, contradicting that C is a hole. Thus, $2 \leq a \leq |H| - 2$, and $uh_a \in E_c$. Let x, y be the next two vertices of C traversed from u^l, h_a, and let P be the y-u^l path in C avoiding h_a and x. If $\texttt{first}(\phi(y)) > a$, then by Proposition 2.5, the length of P is at least two and some inner vertex of P is adjacent to h_a, which contradicts that C is a hole. Hence, $\texttt{first}(\phi(y)) < a$; as a result, $\texttt{last}(\phi(y)) < a$, and by Proposition 2.5, $x \sim h'_{a-1}$. We further traverse C from x, y to the first neighbor v of h'_{a-2}; this vertex exists because $u^l \sim h'_{a-2}$ (noting $a - 2 \geq 0$), and it might be x. We take the hole C (when $v = u^l$ or $v \sim u^l$) or $u^lh'_ax \cdots vh'_{a-2}$ (otherwise). It makes a wheel of $\mho(G)$ with h'_{a-1}, which enables us to call Corollary 3.4.

The second case is when no vertex in C is adjacent to h^r_{-1}. Then C does not intersect R. We argue that C contains no broken pairs. Suppose that $\{x, y\}$ is a broken pair in C with $x \in L$, then $\phi(x) \in T_c \cap T_{cc}$. By property P3, $\texttt{first}(\phi(x)) = -1$, but then by property P2, $y \sim h^r_{-1}$, a contradiction. Therefore, C is free of broken pairs, and by Proposition 3.2, $\phi(C)$ is also a hole of G. We call Lemma 2.2 to verify whether it satisfies property P1. If not, then we have a subgraph of G that is in $\mathcal{F}$ and we are done. If C does not contain any neighbor of h^l_{-1}, then we can return $\phi(C)$ and h_{-1} as a C^*. Otherwise, let x be a neighbor of h^l_{-1} on C, which is in L. By property P3, $\texttt{last}(\phi(x)) = 1$ and both neighbors of x in C are adjacent to h^l_1. Therefore, h_1 has at least three neighbors on the hole $\phi(C)$, and we can use h_1 to replace the inner vertices of $N_{\phi(C)}[h_1]$ to obtain another hole of G. This hole is nonadjacent to h_{-1}, and hence we can return it and h_{-1} as a C^*.

In the remaining cases, $h'_a \notin C$ and C is adjacent to h^r_{-1}. From u^l we traverse C in both directions till the first neighbor(s) of h^r_{-1}, denoted by v, x respectively; note that if h^r_{-1} has only one neighbor in C, then both v and x refer to this vertex. If

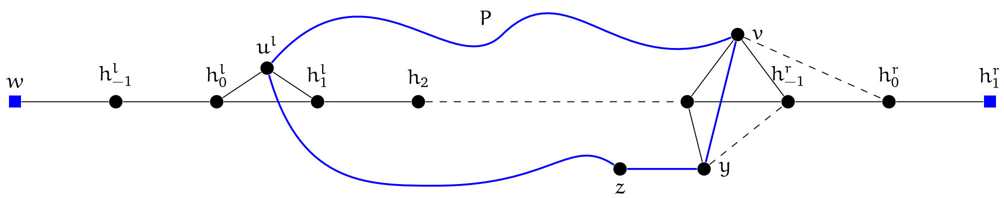


Fig. 12. The last case of the proof of Lemma 3.7.

v and x are distinct and nonadjacent, then we find a wheel of $\mho(G)$ as follows. We find the last neighbor v' of h'_{a-1} from the u^l-v path (avoiding x) in C, and then the first neighbor v'' of h'_{a+1} from the v'-v subpath; their existence is ensured by Proposition 2.5 and they may coincide. Likewise, from the u-x path (avoiding v) in C, we find last neighbor x' of h'_{a-1} and then the first neighbor x'' of h'_{a+1} from the x'-x sub-path. Note that v'' and x'' have to be distinct and nonadjacent, but v' and x' might coincide (when both are u^l) or be adjacent (when one is u^l and the other is not). Depending on the relation between v' and x', either $v' \cdots v''h'_{a+1}x'' \cdots x'h'_{a-1}$ or $v' \cdots v''h'_{a+1}x'' \cdots x'$ is a hole. Every vertex in this hole is adjacent to h'_a, and thus it makes a wheel with h'_a, and we can call Corollary 3.4. Now that $v = x$ or $v \sim x$. We find a sub-path P of C that connects L and v or x as follows. If v or x is adjacent to a vertex in $C \cap L$, then we take this edge as P. Otherwise, P is taken as the path from u^l to v or x that does not contain another vertex from L (recall that $|C \cap L| \leq 2$). Without loss of generality, let it be the u^l-v path. Let y and z be the next two vertices after v in C (note that x is either v or y). The extended path $h^l_0Ph^r_{-1}$ contains a broken pair; hence, either we can call Corollary 3.3 (when there exists a broken pair of distance 2) or find a hole of G with vertices from $\phi(P) \cup \{h_0, h_{-1}\}$; let it be C'. We argue then that C' does not satisfy property P2, and hence we can call Lemma 2.3 to find a subgraph of G that is in $\mathcal{F}$. In particular, we consider the neighborhoods of $\phi(y)$ and $\phi(z)$ on C'; see Fig. 12. By the selection of the path P, the vertex y cannot be in L: the fact $y \in L$ will imply that $v = x$ and is adjacent to both y and u, but this is impossible as C is a hole. For the same reason, $z \neq u$. If $y \in R$, then it cannot be adjacent to h^r_1, as otherwise by properties P3,2, z must be adjacent to h^r_{-1}, a contradiction. As a result, y participates in no broken pairs.

- We have seen $y \notin L$. When $y \in R$, the vertex z can be in neither L (by the construction of $\mho(G)$) nor R (by the assumption that only v and y in C can be adjacent to h^r_{-1}). Thus, $\phi(y)$ and $\phi(z)$ cannot be both adjacent to h_0 in G.
- There cannot be two adjacent vertices of $\mho(G)$ that are adjacent to h^l_{-1} and h^r_{-1} respectively. Otherwise, at least one of properties P1–P3 is invalidated. Therefore, y is not adjacent to h^l_{-1}, while $z \sim h^l_{-1}$ and $y \sim h^r_{-1}$ cannot be both true. Thus, $\phi(y)$ and $\phi(z)$ cannot be both adjacent to h_{-1} in G.
- Since $z \neq u$ and since C is a hole, no inner vertex of P is nonadjacent to y; on the other hand, y participates in no broken pairs. Therefore, $\phi(y)$ can only be adjacent to $\phi(v)$ in $\phi(P)$.

In summary, the neighborhoods of $\phi(y)$ and $\phi(z)$ on $\phi(C)$ are disjoint; hence, $\phi(C)$ does not satisfy property P2. This concludes the proof. □

In the rest $\mho(G)$ will be chordal, and thus we have a minimal non-interval subgraph F of $\mho(G)$ that is chordal. This subgraph is isomorphic to some graph in Fig. 1, on which we use the following notation. It is immediate from Fig. 1 that each of them contains precisely three simplicial vertices (squared vertices), which are called *terminals*, and others (round vertices) are *non-terminal vertices*. It is easy to verify that (a) the distance between a pair of non-terminal vertices is at most 2; and (b) between any pair of terminals there is a path avoiding the closed neighborhood of the other terminal. The vertices in F are labeled as in Fig. 1. To reduce the cases we need to consider in dealing with these graphs, we will heavily exploit their symmetry. Two pairs $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are *symmetrical* in F if there is an automorphism of F that maps x_1 and y_1 to x_2 and y_2 respectively.

- The diameter of a long claw is 4. There are three symmetric pairs of vertices of distance 4, and there are six symmetric pairs of vertices of distance 3.
- The diameter of a whipping top is 3. There are two symmetric pairs of vertices of distance 3.
- The diameter of a net is 3. There are three symmetric pairs of vertices of distance 3.
- The diameter of a † on at least seven vertices is 4. There is only one pair of distance 4. There are four pairs of vertices of distance 3, where $\{t_1, t_3\}$ and $\{t_2, t_3\}$ are symmetric, while $\{t_1, u_2\}$ and $\{t_2, u_1\}$ are symmetric.

Lemma 3.8. *Let $\mho(G)$ be chordal. Given a subgraph F of $\mho(G)$ in Fig. 1, we can in $O(n+m)$ time find a subgraph of G that is in $\mathcal{F}$.*

Proof. Recall that w is simplicial in $\mho(G)$. If w is contained in F, then it is simplicial in F as well, which means that it is a terminal of F, and has at most two neighbors in F. We consider first the case that (a) w has two distinct neighbors u^l and v^l in F (note then $u, v \in T_{cc}$), and (b) no vertex in $\overline{T}$ is adjacent to both u and v in G. By the definition of T_{cc}, we can find two distinct vertices $x, y \in \overline{T}$ such that $ux, vy \in E_{cc}$. By assumption, $u \not\sim y$ and $v \not\sim x$ in G. As a result, x and y are nonadjacent in G; otherwise, u^rv^ryx is a hole of $\mho(G)$, which contradicts the assumption that $\mho(G)$ is chordal. We apply the procedure described in Fig. 13.

We now verify the correctness of the procedure. Both u and v are adjacent h_{-1} in G, and hence according to property P3, $\texttt{last}(u)$ and $\texttt{last}(v)$ are either 0 or 1. By property P2, $x \not\sim h_{\texttt{last}(u)}$. Step 1 considers the case when $x \sim h_{\texttt{last}(u)+1}$, then

1. **if** $\mathtt{last}(u) = 1$ and $x \sim h_2$ **then** $\quad\backslash\!\backslash\ 0 \leqslant \mathtt{last}(u), \mathtt{last}(v) \leqslant 1.$
    call Lemma 2.3 with hole xuh_1h_2 and **return** the found subgraph in $\mathcal{F}$;
   if $\mathtt{last}(u) = 0$ and $x \sim h_1$ **then**
    call Lemma 2.3 with hole xuh_0h_1 and **return** the found subgraph in $\mathcal{F}$;
   if $y \sim h_{\mathtt{last}(v)+1}$ **then** symmetric as above;
2. **if** $\mathtt{last}(u) = \mathtt{last}(v)$ **then**
    return $\{x, u, y, v, h_{\mathtt{last}(v)}, h_{\mathtt{last}(v)+1}\}$ as a †; $\quad\backslash\!\backslash$ *Fig. 14a.*
   $\backslash\!\backslash$ assume from now that $\mathtt{last}(u) = 1$ and $\mathtt{last}(v) = 0$.
3. **if** $v \sim h_2$ **then return** $vh_0h_1h_2$ and x as a C^*;
4. **if** $y_2 \not\sim h_{-1}$ **then return** $\{x, h_{-1}, y, v, h_0, h_1\}$ as a †; $\quad\backslash\!\backslash$ *Fig. 14b.*
   else return $\{x, h_{-1}, u, y, v, h_0, h_1\}$ as a ‡. $\quad\backslash\!\backslash$ *Fig. 14c.*

Fig. 13. Procedure for Lemma 3.8.

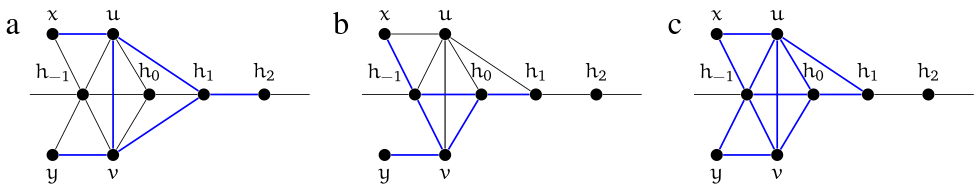


Fig. 14. Structures used in the proof of Lemma 3.8 (not all edges of the graph are presented).

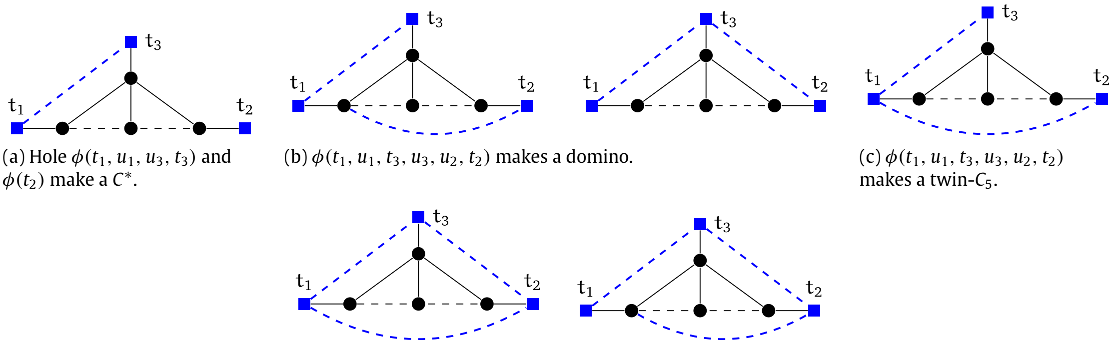
xuh_0h_1 or xuh_1h_2 is a hole of G, depending on $\mathtt{last}(u)$ is 0 or 1. On the hole xuh_1h_2, only u and h_1 can be adjacent to v, and they are nonadjacent to y; therefore, the hole does not satisfy property P2. Likewise, on the hole xuh_0h_1, only u and h_0 can be adjacent to v but not y, while h_1 can be adjacent to only one of v and y. Thus, Lemma 2.3 will find a subgraph in $\mathcal{F}$. The case when $y \sim h_{\mathtt{last}(v)+1}$ can be dealt with in a similar way. Now that $x \not\sim h_{\mathtt{last}(u)+1}$ and $y \not\sim h_{\mathtt{last}(v)+1}$, if $\mathtt{last}(u) = \mathtt{last}(v)$, then $\{x, u, y, v, h_{\mathtt{last}(v)}, h_{\mathtt{last}(v)+1}\}$ induces a net; this justifies step 2. If the procedure has passed step 2, then $\mathtt{last}(u)$ must be different from $\mathtt{last}(v)$; recall that they have to be either 0 or 1, and hence there are only two possible cases. Steps 3 and 4 deal with the case $\mathtt{last}(u) = 1$ and $\mathtt{last}(v) = 0$, and the other case can be dealt with in a symmetric way. If $v \sim h_2$, then $vh_0h_1h_2$ is a hole, and no vertex on it is adjacent to x; this justifies step 3. For step 4, note that by properties P3, $\mathtt{first}(u) = -1$, and by the definition of E_{cc} and property P2, $x \sim h_{-1}$. This concludes the correctness of the procedure (see Fig. 14).

In the remaining cases, if $w \in F$ and $|N_F(w)| = 2$, then we can find a vertex $w' \in \overline{T}$ such that w' is adjacent to v^r for both $v^l \in N_F(w)$. By construction, w' cannot be adjacent to $N_F(w)$, and thus the distance between w and w' is at least 2. We extend ϕ by defining $\phi(w) = w'$. If w' is adjacent to some vertex x in F (it must be the case when $w' \in F$), then $\{w, x\}$ can be viewed similar as a broken pair: they are not adjacent in $\mho(G)$ but $\phi(w)$ and $\phi(x)$ are adjacent in G.

If F contains neither a broken pair nor any neighbor of w', then $\phi(F)$ is isomorphic to F and we are done. Therefore, we assume at least one of them exists. We consider the shortest distance of a broken pair in F, or the shortest distance between w and $N_F(w')$ in F; let d be the smaller of them. Note that given such a path of length d in F, we can find an u^l-u^r path of length $d+1$ for some vertex $u \in T$. In particular, let u^l be the second vertex of a w-x path of length d, where $x \sim w'$, then the path can be obtained by removing w and appending w', u^r after x. By the construction of $\mho(G)$, we have $d > 1$. If $d = 2$, then there is a u^l-u^r path P of length 3, which enables us to call Corollary 3.3. On the other hand, the diameter of the subgraph F is at most four. Therefore, in the following we assume that d is either 3 or 4. If there is a broken pair $\{x, y\}$, then one of x and y must be a terminal of F; we may assume that x is a terminal of F. It should be noted that w' itself might be in F, just as F may contain both u^l and u^r for some $u \in T$; this can only happen when $d = 3$ and the diameter of F is 4.

If F is a long claw, then its four non-terminal vertices are derived from a claw of G. The other three vertices are nonadjacent to c; thus, we can call Proposition 3.5(I). If F is a whipping top, then $\phi(F)$ has the same cardinality as F. There might be one or two edges $\phi(t_1)\,\phi(t_3)$ and/or $\phi(t_2)\,\phi(t_3)$. We have a domino and a C_4^* respectively. If F is a ‡, then F has at least seven vertices (the diameter of a tent is 2). The pair must be $\{t_1, t_2\}$. Let P be the t_1-t_2 path in $F - N[t_3]$. We can return $\phi(P)$ and $\phi(t_3)$ as a C^*. If F is a net, then its three non-terminal vertices are derived from a triangle of G, each having a distinct neighbor. Thus, we can call Proposition 3.5(II). In the rest, F is a † on at least seven vertices. If $d = 3$, then we consider the pairs $\{t_1, t_3\}$ and $\{t_1, u_2\}$. (Other cases are handled in a similar way and are omitted.) First, let it be $\{t_1, t_3\}$, and let P be the path $t_1u_1u_3t_3$. Based on the adjacency between $\phi(t_2)$ and vertices in $\phi(P)$, one of the following holds true.

- If $\phi(t_2)$ is nonadjacent to the hole induced by $\phi(P)$, then we return $\phi(P)$ and $\phi(t_2)$ as a C^* (Fig. 15(a)).
- If $\phi(t_2)$ is adjacent to only $\phi(t_3)$ or $\phi(u_1)$, then we get a domino (Fig. 15(b)).
- If $\phi(t_2)$ is adjacent to only $\phi(t_1)$, then we get a twin-C_5 (Fig. 15(c)).
- If $\phi(t_2)$ is adjacent to $\phi(t_1)$ and precisely one of $\{\phi(t_3), \phi(u_1)\}$, then we get an FIS-1 (Fig. 15(d)).
- If $\phi(t_2)$ is adjacent to both $\phi(t_3)$ and $\phi(u_1)$, then we get a $K_{2,3}$ (Fig. 15(e)). Here the adjacency between $\phi(t_2)$ and $\phi(t_1)$ is immaterial.



(a) Hole $\phi(t_1, u_1, u_3, t_3)$ and $\phi(t_2)$ make a C^*.

(b) $\phi(t_1, u_1, t_3, u_3, u_2, t_2)$ makes a domino.

(c) $\phi(t_1, u_1, t_3, u_3, u_2, t_2)$ makes a twin-C_5.

(d) $\phi(t_1, u_1, t_3, u_3, u_2, t_2)$ makes an FIS-1.

(e) $\phi(u_1, t_3, u_3, u_2, t_2)$ makes a $K_{2,3}$ (t_1 is irrelevant here).

Fig. 15. F is a † on at least 7 vertices (dashed edges are in G only).

Second, consider the pair $\{t_1, u_2\}$. If $\phi(t_1)\phi(t_3) \in E(G)$ as well, then $\phi(\{t_1, u_1, u_3, t_3, u_2\})$ makes a $K_{2,3}$; otherwise, let P be the t_1-u_2 path in $F - N[t_3]$, we can return $\phi(P)$ and $\phi(t_3)$ as a C^*. Now $d = 4$, $\phi(F)$ has the same number of vertices as F. The only pair is $\{t_1, t_2\}$, and there is thus a C^*. □

4. A recognition algorithm for proper Helly circular-arc graphs

Proper Helly circular-arc graphs are a subset of normal Helly circular-arc graphs: if a circular-arc model is both proper and Helly, then it is normal as well. Indeed, a normal Helly circular-arc graph is a proper Helly circular-arc graph if and only if it is claw-free. Therefore, for the recognition of proper Helly circular-arc graphs, we may run the algorithm presented in previous sections first, and in the case that the input graph is a normal Helly circular-arc graph, check whether it contains a claw. But there is yet a simpler way that does not need to call the recognition algorithm for interval graphs.

We start from recalling the forbidden subgraph characterization of proper Helly circular-arc graphs.

Theorem 4.1 ([15,23]). *A graph is a proper Helly circular-arc graph if and only if it contains no claw, net, tent, W_4, W_5, $\overline{C_6}$, or C^*_ℓ.*

A technical remark is worth here. What was characterized by Lin et al. [15, Corollary 5] is actually the proper circular-arc graphs that are not proper Helly circular-arc graphs: they must contain a W_4 or tent. On the other hand, Tucker [23] had characterized the forbidden induced subgraphs of proper circular-arc graphs, which include, aside from those stated in Theorem 4.1 (claw, net, W_5, $\overline{C_6}$, and C^*_ℓ for $\ell \geq 4$), $\overline{C_{2\ell}}$ and $\overline{C^*_{2\ell-1}}$ for $\ell \geq 4$. To see Theorem 4.1, note that for $\ell \geq 4$, both $C_{2\ell}$ and $C^*_{2\ell-1}$ contain a $\overline{W_4}$. Let $\mathcal{F}_P$ denote the set of claw, net, tent, W_4, W_5, $\overline{C_6}$, and C^*_ℓ. A quick glance tells us that any graph in $\mathcal{F}$ but not in $\mathcal{F}_P$ contains a claw. Therefore, given a subgraph of G in $\mathcal{F}$, it is straightforward to retrieve a subgraph of G in $\mathcal{F}_P$.

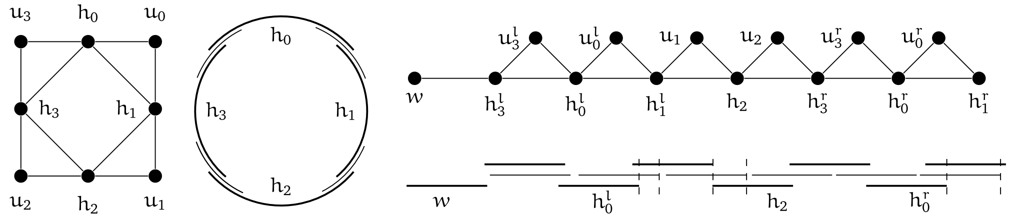
Since proper interval graphs are {claw, net, tent, C_ℓ}-free graphs [21,25], it follows from Theorem 4.1 that if a proper Helly circular-arc graph is chordal, then it is a proper interval graph. Therefore, for the recognition of proper Helly circular-arc graphs, we are also focused on non-chordal graphs. We find a hole that satisfies properties P1–P3, and build the auxiliary graph $\mho(G)$. It is easy to verify that $\mho(G)$ contains a claw if and only if G contains a claw. Thus, if the graph G has passed previous test, its recognition reduces to the recognition of a proper interval graph. To make it certifying, we adapt Theorems 1.3 and 2.7 as follows. We use the well-known fact that every proper interval graph has a unit interval model as well [21].

Theorem 4.2. *Let $\mho(G)$ be the auxiliary graph defined with a hole H satisfying properties* P1–P3. *If $\mho(G)$ is a proper interval graph, then we can in $O(n + m)$ time build a circular-arc model for G that is normal and proper.*

Proof. We can in $O(n + m)$ time build a unit interval model $\mathcal{I}$ for $\mho(G)$. Let $0 = \mathtt{rp}(w)$ and $a = \max_{u \in \overline{T}} \mathtt{rp}(u)$. Without loss of generality, assume the path P_H goes "from left to right" in $\mathcal{I}$. Then $\mathtt{lp}(h_0^l) > 0$ and $a > \mathtt{rp}(h_0^l) > 1$. We modify $\mathcal{I}$ such that for each pair of vertices $u, v \in T_{cc}$,

$$\mathtt{lp}(u^l) < \mathtt{lp}(v^l) \quad \text{if and only if} \quad \mathtt{lp}(u^r) < \mathtt{lp}(v^r). \tag{4}$$

Let u, v be pair of vertices in T_{cc} such that $\mathtt{lp}(u^r) < \mathtt{lp}(v^r)$ but $\mathtt{lp}(u^l) > \mathtt{lp}(v^l)$. We argue that (i) there cannot be any interval with right endpoint in $[\mathtt{lp}(v^l), \mathtt{lp}(u^l)]$, and (ii) there cannot be any interval with left endpoint in $[\mathtt{rp}(v^l), \mathtt{rp}(u^l)]$. Suppose to the contradiction of (i), $\mathtt{lp}(v^l) < \mathtt{rp}(x) < \mathtt{lp}(u^l)$ for some x. If $x = w$, then there must be some vertex $y \in \overline{T}$ such that $yv \in E_{cc}$. But then y is adjacent to v^r but not u^r in $\mho(G)$, and it is impossible that $\mathtt{lp}(u^r) < \mathtt{lp}(v^r)$. Otherwise, $x \in L$, and there is a corresponding vertex $x' \in R$. Then $\phi(x)$ is adjacent to h_{-1}, v but not u, and it is again impossible that



(a) A proper Helly circular-arc graph. (b) $\mho(G)$ and a unit interval model.

Fig. 16. A proper Helly circular-arc graph G and a unit interval model for $\mho(G)$. Since $\mathtt{rp}(h^l_1) - \mathtt{rp}(h^l_0) < 1 < \mathtt{rp}(u_1) - \mathtt{rp}(u^l_0)$, it follows that $\mathtt{rp}(u^l_0) - \mathtt{rp}(h^l_0) < \mathtt{rp}(u_1) - \mathtt{rp}(h^l_1)$. By repeating this argument we can conclude $\mathtt{rp}(u^l_1) - \mathtt{rp}(h^l_0) < \mathtt{rp}(u^r_1) - \mathtt{rp}(h^r_0)$.

INPUT: a graph G.
OUTPUT: either a proper and Helly circular-arc model for G or a subgraph of G that is in $\mathcal{F}_P$.

1. **if** G is chordal **then**
   return either a proper interval model for G or a claw, net, tent;
2. find a hole H;
3. **if** H does not satisfy any of P1–P3 **then return** a subgraph of G that is in $\mathcal{F}_P$;
4. call the algorithm of Lemma 2.4 to construct the auxiliary grpah $\mho(G)$;
5. **if** $\mho(G)$ is not a proper interval graph **then**
   call Lemma 4.3 and **return** the found subgraph of G that is in $\mathcal{F}_P$;
6. **call** Theorem 2.7 to build a normal and proper circular-arc model $\mathcal{A}$ for G;
7. **if** $\mathcal{A}$ is Helly **then return** $\mathcal{A}$;
   else return a W_4, W_5, or claw (Lemma 2.8).

Fig. 17. The recognition algorithm for proper Helly circular-arc graphs.

$\mathtt{lp}(u^r) < \mathtt{lp}(v^r)$. The case (ii) can be refuted in a similar way. Therefore, we can switch $I(u^l)$ and $I(v^l)$. Repeating this we can find a modified model that satisfies (4) and represents $\mho(G)$.

We use $\mathcal{I}$ to construct a set of arcs for $V(G)$ on a circle of perimeter $a + \epsilon$ (where ϵ is a small positive number such that no endpoint of $\mathcal{I}$ lies in $(a, a + \epsilon]$) as follows:

$$A(v) := \begin{cases} [\mathtt{lp}(v^r), \mathtt{rp}(v^l)] & \text{if } v \in T_{cc}, \\ I(v^l) & \text{if } v \in T \setminus T_{cc}, \\ I(v) & v \in \overline{T}. \end{cases} \tag{5}$$

The construction is the same as that used in Theorem 4.2. A word-by-word copy of the proof in Theorem 4.2 will show that $\mathcal{A}$ is a normal circular-arc model for G. We now verify $\mathcal{A}$ is proper, i.e., no arc in it contains another arc. Let u, v be any two vertices of G. First, if both are in T_{cc}, then it follows from (4). If neither is in T_{cc}, then $A(u) = I(u)$ and $A(v) = I(v)$. Otherwise, assume that u is in T_{cc} and v is not. Since $A(v)$ does not contain 0, it cannot contains $A(u)$; on the other hand, since both $[\mathtt{lp}(u^r), b)$ and $[0, \mathtt{rp}(u^l)]$ is shorter than the unit length, $A(u)$ cannot contain $A(v)$ that has length 1. □

One may wonder whether we can always arrange the intervals such that for each $v \in T$, the intervals $I(v^l)$ and $I(v^r)$ can have the same position related to $I(h^l_0)$ and $I(h^r_0)$ respectively. The answer is no. See, for example, Fig. 16. From such an interval model, if it would exist, the circular-arc model constructed by Theorem 4.1 should always be a unit circular-arc model. But we know that some proper (Helly) circular-arc graphs have no unit (and Helly) circular-arc models. Indeed, the eight-vertex proper Helly circular-arc graph G given in Fig. 16(a) is actually the CI(4, 1) graph defined by Tucker [23], which is not a unit circular-arc graph; see also [15].

Lemma 4.3. *Given a minimal subgraph of $\mho(G)$ that is not a proper interval subgraph, we can in $O(n + m)$ time find a subgraph of G in $\mathcal{F}_P$.*

Proof. Let F be the subgraph of $\mho(G)$. If F is a claw, then we call Corollary 3.4; otherwise we call the algorithm of Theorem 1.3. Either the returned subgraph is in $\mathcal{F}_P$, or we can find a claw from it. □

We point out that Lemma 4.3 can be argued in a direct way, without calling the algorithm of Theorem 1.3, and it is far simpler than what we have done in Section 3. On the one hand, there cannot be a vertex in both T_c and T_{cc}. On the other hand, we do not need take care of long claws, †s, and ‡s.

We are now ready to present the certifying recognition algorithm for proper Helly circular-arc graphs in Fig. 17, from which Theorem 1.4 follows.

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