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Weak harmonic labeling of graphs and multigraphs ☆

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ABSTRACT

In this article we introduce the notion of *weak harmonic labeling* of a graph, a generalization of the concept of harmonic labeling defined recently by Benjamini, Cyr, Procaccia and Tessler that allows extension to finite graphs and graphs with leaves. We present various families of examples and provide several constructions that extend a given weak harmonic labeling to larger graphs. In particular, we use finite weak models to produce new examples of (strong) harmonic labelings. As a main result, we provide a characterization of weakly labeled graphs in terms of *harmonic subsets* of $\mathbb{Z}$ and exhibit quantitative evidence of the efficiency of this method for computing all weakly labeled finite graphs as opposed to an exhaustive search calculation. In particular, we characterize harmonically labeled graphs as defined by Benjamini et al. We further extend the definitions and main results to the case of multigraphs and total labelings.

1. Introduction

The notion of harmonic labeling of an infinite (simple) graph was introduced recently by Benjamini, Cyr, Procaccia and Tessler in [1]. If $G = (V, E)$ is an infinite graph of bounded degree then an *harmonic labeling* of G is a bijective function $\ell : V \to \mathbb{Z}$ such that

$$\ell(v) = \frac{1}{\deg(v)} \sum_{\{v,w\} \in E} \ell(w) \tag{1}$$

for every $v \in V$. In [1] the authors provide some examples of harmonic labelings and prove the existence of such labelings for regular trees and the lattices $\mathbb{Z}^d$ and the non-existence for cylinders $G \times \mathbb{Z}$ for non-trivial G. Graph labeling is a widely developed topic and has a broad range of applications (see, e.g., [2–4]).

Harmonically labelable graphs seem to have a rather restrictive configuration. Particularly, these graphs do not have leaves (vertices of degree 1) since there are no one to one functions verifying harmonicity on such vertices. This actually turns out to be the main obstacle for a generalization of this concept to the context of finite graphs, which is a natural extension taking into account the fruitful link between harmonic functions and geometric properties of finite graphs (see e.g. [5, §4]). Furthermore, finite examples might be useful as local models to produce new harmonically labeled (infinite) graphs.

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In this paper we propose a two-way generalization of the notion of harmonic labeling, introducing the concept of *weak harmonic labeling*. On one hand, we require satisfying equation (1) only for $v \in V \setminus S$, where S is the set of leaves of G. On the other hand, we let the function ℓ be a bijection with an integer interval I (finite or infinite). These conditions permit a straightforward extension of harmonic labelings to the finite setting. This results in a more general structure which provides a far wider theory, which was one of the ambitions in [1].

We present several examples of weak harmonic labelings and show the non-existence of this type of labelings for various families of (finite and infinite) graphs. We further introduce constructions to obtain new examples from given ones. In particular, we define the notion of inner cylinder and a way to extend any weakly labeled finite graph into an infinite one. We use weak finite models to construct new families of harmonic labelings. In particular, we exhibit a non-numerable collection of harmonically labeled graphs, which additionally contains an infinite number of examples spanned by finite sets of vertices. This answers a question raised in [1] about the existence of connected graphs different from $\mathbb{Z}$ which admit an harmonic labeling spanned by a finite set (see Remark 3.6).

The main result of this article is the characterization of weakly labeled graphs in terms of certain families of collections of finite subsets of $\mathbb{Z}$ called *harmonic subsets*. Since the statement of this result without many preliminary conventions would be too lengthy, the reader is invited to turn to Lemma 4.3 and Theorem 4.6 for a first impression. In particular, we obtain a characterization of harmonically labeled graphs (as defined in [1]) in terms of the aforementioned harmonic subsets (Corollary 4.7). The method to compute all weakly labeled graphs of n vertices derived from this characterization is far more efficient than testing the number of all possible n-vertex graphs $G = (V, E)$ and all possible bijective maps of an integer interval of size n onto V (see Section 5). This technique is used to find all weakly labeled graphs of up to (and including) ten vertices.

All the definitions and results of weak harmonic labelings can be extended to the case of multigraphs (or total labelings) in a straightforward way. We prove the version for multigraphs of Theorem 4.6 and exhibit an algorithm that produces a total weak harmonic labeling from a given admissible labeling (see Algorithm 5).

The paper is organized as follows. In Section 2 we introduce the concept of harmonic labeling and exhibit several examples of (families) of weakly labeled (finite and infinite) graphs. In Section 3 we present two constructions to obtain a new labelings from a given one and we use finite models of weakly labeled graphs to construct new families of harmonically labeled graphs. In Section 4 we prove the characterization of weakly labeled graphs (and, in particular, of harmonic labelings) in terms of families of collections of *harmonic subsets of* $\mathbb{Z}$. In Section 5 we provide quantitative evidence of the computational advantage of using the main characterization to find weakly labeled graphs and exhibit the list of all possible weakly labeled graphs up to (and including) ten vertices. In Section 6 we extend the definitions and main results of the theory to the case of multigraphs and total labelings. Finally, Section 7 proposes a list of open problems and future directions for this theory.

2. Weak harmonic labelings of simple graphs

All graphs considered have bounded degree and connected components of at least three vertices (see Remark 2.6). For a simple graph G we write V_G for its set of vertices and E_G for its set of edges. We put $v \sim w$ if v and w are adjacent and we let $N_G(v) = \{v\} \cup \{w : w \sim v\} \subset V_G$ denote the closed neighborhood of v. Throughout, S_G will denote the set of leaves (vertices of degree 1) of G and I will denote a generic integer interval (a set of consecutive integers).

Remark 2.1. Note from the above considerations that, for any G, $v \sim w$ implies $\{v, w\} \cap (V_G \setminus S_G) \neq \emptyset$.

Definition 2.2. A *weak harmonic labeling* of a graph G (simply *weak labeling* in this context) is a bijective function $\ell : V_G \to I$ such that

$$\ell(v) = \frac{1}{deg(v)} \sum_{w \sim v} \ell(w) \qquad \forall v \in V_G \setminus S_G. \tag{2}$$

When we want to explicitate the interval of the labeling, we shall refer to it as a *weak harmonic labeling onto* I.

As mentioned earlier, the relativeness to $V_G \setminus S_G$ of the harmonicity property is natural as there cannot be one to one functions with harmonic leaves. Harmonic labelings are particular cases of weak harmonic labelings since harmonically labelable infinite graphs have no leaves. More precisely, a weak harmonic labeling onto I is an harmonic labeling if and only if $I = \mathbb{Z}$ and $S_G = \emptyset$.

Remark and Convention 2.3. Since a function ℓ satisfies equation (2) if and only if $\pm\ell + k$ satisfies it for any $k \in \mathbb{Z}$, we shall not distinguish between labelings obtained from translations or invertions. Thus, we make the convention that in the case $I \neq \mathbb{Z}$ we shall normalize all labelings to the intervals $[0, |V_G| - 1] \cap \mathbb{Z} = \{k \in \mathbb{Z} : 0 \leq k \leq |V_G| - 1\}$ or $[0, \infty) \cap \mathbb{Z} = \{k \in \mathbb{Z} : k \geq 0\}$.

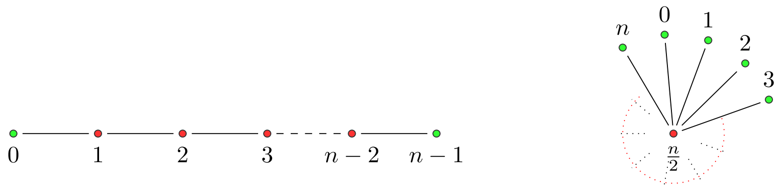

Fig. 1. Weak harmonic labeling on P_n and $K_{1,n}$.

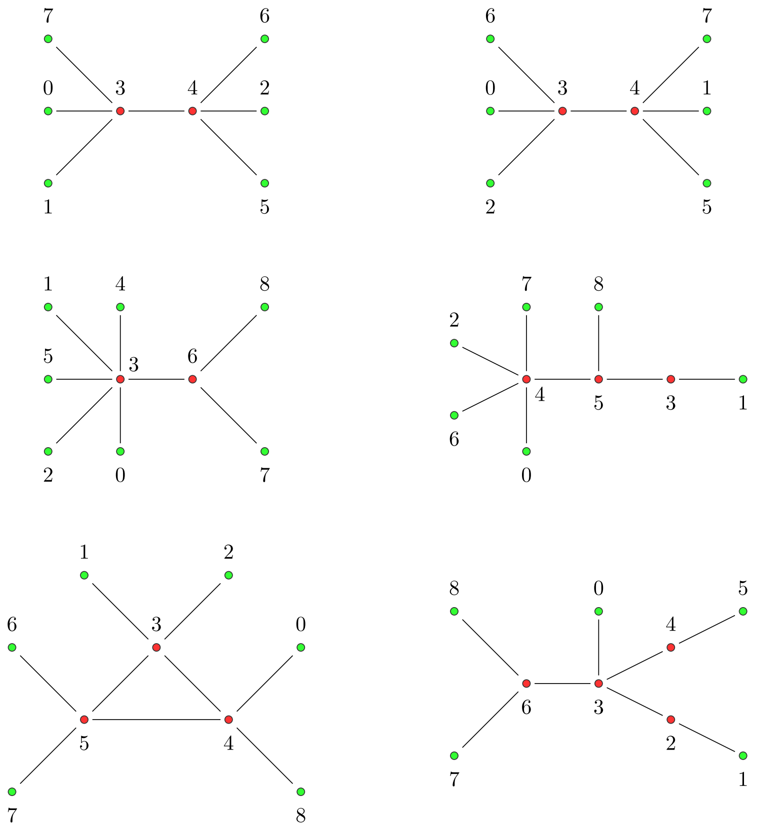

Fig. 2. Some examples of weakly labeled finite (connected) graphs (taken from Table 4 in Section 5).

The simplest examples of weakly labeled finite graphs are the paths P_n and the stars $K_{1,n}$ for even n (Fig. 1). Paths can be extended either to ∞ or to both $-\infty$ and ∞ to obtain a weak harmonic labeling onto $[0,\infty)\cap\mathbb{Z}$ or $\mathbb{Z}$ respectively. In the latter, we obtain the trivial harmonically labeled graph $\mathbb{Z}$. More examples of weakly labeled finite graphs are shown in Figs. 2 and 3, where particularly it can be verified that a given graph can admit more than one weak harmonic labeling (see e.g. Fig. 2 (top)). Note that P_n and $K_{1,n}$ (n even) are extremal cases of the collection pictured in Fig. 3 (top). The non-acyclic family in Fig. 3 (bottom), which can be inferred from the examples computed in Section 5, can be trivially extended to labelings onto $[0,\infty)\cap\mathbb{Z}$ and $\mathbb{Z}$. In the latter, we obtain again an harmonic labeling. Furthermore, adding the edges $\{\{2k-1,2k+1\}\,|\,k\in\mathbb{Z}\}$ produces another such labeling. These two examples are different from all those presented in [1], which evidences how new examples of harmonic labelings can be deduced from finite weakly labeled ones. We shall present more examples obtained in this fashion in the next section.

Note that the minimum and maximum values of a weak harmonic labeling over a finite G must take place on leaves, so any finite graph with less than two leaves does not admit weak harmonic labelings. This is the analogue result that non-constant harmonic functions have at least two poles (see e.g. [5, §4]). In particular, cycles, complete graphs K_n with $n\geq 3$, complete bipartite graphs $K_{n,m}$ with $n,m\geq 2$ and cylinders $G\times P_n$ for $n\geq 2$ and any G do not admit a weak harmonic labeling. It is not hard to characterize finite graphs with maximum and minimum number of leaves which admit this type of labeling.

Lemma 2.4. *Let G be an n-vertex graph which admits a weak harmonic labeling.*

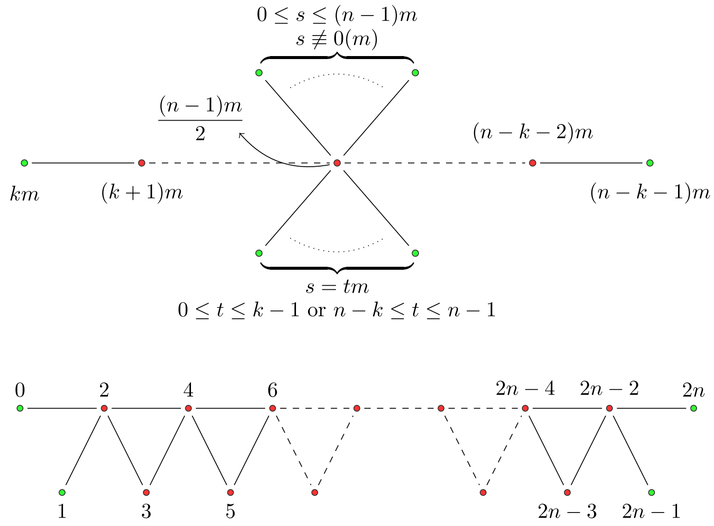

Fig. 3. Top: A collection of weakly labeled graphs for $m, n, k \in \mathbb{Z}_{\geq 0}$, $m \geq 1$, n odd and $0 \leq k \leq \frac{(n-1)m}{2}$. The graphs P_n and $K_{1,n}$ (n even) are extremal cases of this family for $m = 1$. **Bottom:** A family of weakly labeled finite graphs than can additionally be extended to weakly labeled graphs onto $[0, \infty) \cap \mathbb{Z}$ and $\mathbb{Z}$.

(1) G has two leaves if and only if $G = P_n$.
(2) G has $n-1$ leaves if and only if n is even and $G = K_{1,n}$.

Proof. We prove the direction of (1), which is the only non-trivial implication. Let $\ell : V_G \to I$ be a weak harmonic labeling of G and denote by v_i the vertex labeled i. We may assume $n \geq 4$. By the remarks in the previous paragraph about the maximum and minimum values of weak labelings, v_0 and v_n are the leaves of G. Since the vertex $v_1 \notin S_G$ then $v_1 \sim v_0$. Now

$$\deg(v_1) = \sum_{w \sim v_1} \ell(w) \geq \sum_{\substack{w \sim v_1 \\ w \neq v_0}} 2 = 2(\deg(v_1) - 1),$$

from where $\deg(v_1) = 2$. Therefore, $N_G(v_1) = \{v_0, v_2\}$. The same argument shows that $N_G(v_{n-1}) = \{v_{n-2}, v_n\}$. Assume inductively that $N_G(v_i) = \{v_{i-1}, v_{i+1}\}$ for $0 < i < k < n-1$. Then

$$\deg(v_k)k = \sum_{w \sim v_k} \ell(w) \geq (k-1) + \sum_{\substack{w \sim v_k \\ w \neq v_{k-1}}} (k+1) = k - 1 + (k+1)(\deg(v_k) - 1),$$

and $\deg(v_k) \leq 2$. This proves that $N_G(v_k) = \{v_{k-1}, v_{k+1}\}$ and hence $G = P_n$. $\square$

Remark 2.5. Recall that the *Laplacian* of a finite graph G is the operator $L_G = D - A \in \mathbb{Z}^{n \times n}$ where A is the adjacency matrix of G and D is the diagonal degree matrix. If we let $\tilde{L}_G$ denote the operator obtained from L_G by removing the rows corresponding to leaves (the *reduced Laplacian* of G) then G admits a weak harmonic labeling if and only if there exists a permutation $\sigma \in S_n$ such that $\sigma(0, \ldots, n-1) \in ker(\tilde{L}_G)$.

Remark 2.6. The case of general graphs (dropping the restriction of connected components of more than two vertices) gives rise to uninteresting examples as these components are “invisible” to the requirement of harmonicity and can be used to complete partial one-to-one labelings. Note however that, even with our initial requirements, weakly labeled non-connected graphs have many superfluous examples, trivially built from connected ones. For example, given a finite graph G and a weak labeling $\ell : V_G \to [0, n-1] \cap \mathbb{Z}$, if we let $H = \bigvee_{1 \leq i \leq k} G_i$ be the disjoint union of $k \in \mathbb{N}$ copies of G then we can define a weak harmonic labeling $\ell_H : H \to [0, kn-1] \cap \mathbb{Z}$ as follows:

$$\ell_H(v) = \ell(v) + (i-1)n, \text{ if } v \in V_{G_i}.$$

3. Harmonic labelings from finite weak models

More complex weakly labeled (finite and infinite) graphs can be built up from simpler finite examples. Some of these graphs can be inferred from the structure of the finite model and some can be constructed by performing unions and considering cylinders on them. In many cases, we shall obtain (new) harmonically labeled graphs.

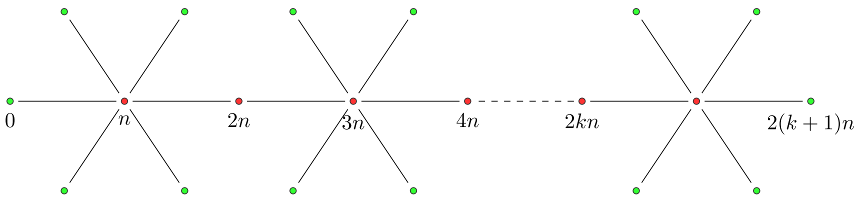

Fig. 4. Extending weak harmonic labelings through coalescence.

3.1. Coalescence and inner cylinders

We first show two constructions that produce new weakly labeled (finite and infinite) graphs from a finite weak model. Particularly, these constructions provide a way to produce infinitely many weak harmonic labelings onto $[0,\infty)\cap\mathbb{Z}$ and $\mathbb{Z}$.

For simple graphs G, H and $v \in V_G$ and $w \in V_H$ we let $G \cdot_v^w H$ denote the graph obtained from $G \cup H$ by identifying the vertex v with the vertex w (this is sometimes referred by some authors as the *coalescence* between G and H at vertices v and w).

Lemma 3.1. *Let $\ell_G : V_G \to [0, n-1]\cap\mathbb{Z}$ and $\ell_H : V_H \to I$, $I = [0, m-1]\cap\mathbb{Z}$ or $[0,\infty)\cap\mathbb{Z}$ be weak harmonic labelings on graphs G and H respectively. Let $v_i \in V_G$ be the vertex labeled i in G $(0 \le i \le n-1)$ and $w_j \in V_H$ be the vertex labeled j in H $(0 \le j \le m-1)$. If the sole vertex v adjacent to v_{n-1} in G and the sole vertex w adjacent to w_0 in H satisfy $\ell_G(v) + \ell_H(w) = n-1$ then there exists a weak harmonic labeling of $G \cdot_{v_{n-1}}^{w_0} H$.*

Proof. The desired weak harmonic labeling ℓ over $G \cdot_{v_{n-1}}^{w_0} H$ is given

$$\ell(u) = \begin{cases} \ell_G(u) & u \in G \\ \ell_H(u) + n - 1 & u \in H. \end{cases} \quad \square$$

Fig. 4 shows a particular example of extending a weakly labeled graph by coalescence.

Note that the construction of Lemma 3.1 can be iterated to produce infinitely many new examples (both finite and infinite). Furthermore, *any* weakly labeled graph can be extended to a new (finite or infinite) weakly labeled graph since the family of bipartite complete graphs $\{K_{1,n} : n \text{ even}\}$ has a member of average k for each $k \in \mathbb{N}$. In some cases, we can "complete" these infinite weakly labeled examples to harmonic labelings. Fig. 5 shows three harmonically labeled graphs produced from the coalescence of infinite copies of $K_{1,n}$ for $n = 4, 6, 8$.

The other aforementioned construction, which produces exclusively weak harmonic labelings onto $\mathbb{Z}$, is based on the notion of *inner cylinder* of a graph.

Definition 3.2. Given a graph G, we define the *inner cylinder of* G as the graph $G \check{\times} \mathbb{Z}$ such that:

- $V_{G\check{\times}\mathbb{Z}} = \{(v,i) : v \in V_G, i \in \mathbb{Z}\}$
- $(v,i) \sim (w,j)$ if and only if ($i = j$ and $\{v,w\} \in E_G$) or ($v = w \in V_G \setminus S_G$ and $i = j+1$ or $i = j-1$).

Interestingly, examples of weak harmonic labelings onto $\mathbb{Z}$ can be produced from *any* finite example as the following lemma shows.

Lemma 3.3. *A weak harmonic labeling on a finite graph G induces a weak harmonic labeling onto $\mathbb{Z}$ on $G \check{\times} \mathbb{Z}$.*

Proof. Write $|V_G| = n$ and let $\ell : V_G \to [0, n-1]\cap\mathbb{Z}$ be a weak harmonic labeling. Then, the claimed labeling $\ell' : V_{G\check{\times}\mathbb{Z}} \to \mathbb{Z}$ over $G \check{\times} \mathbb{Z}$ is given by

$$\ell'(v,k) = \ell(v) + kn. \quad \square$$

Fig. 6 (top) shows examples of weak harmonic labelings onto $\mathbb{Z}$ defined using this construction. Similarly to the case of coalescence, in some cases we can extend these weak infinite examples to harmonic labelings. For instance, the weak harmonic labelings of $K_{1,2} \check{\times} \mathbb{Z}$ and $K_{1,4} \check{\times} \mathbb{Z}$ given in Lemma 3.3 can be extended to the harmonic labelings shown in Fig. 6 (bottom). Note that we have produced two new harmonically labeled graphs from the (same) weak labeling of $K_{1,4}$; namely, Fig. 5 (top) and Fig. 6 (bottom right).

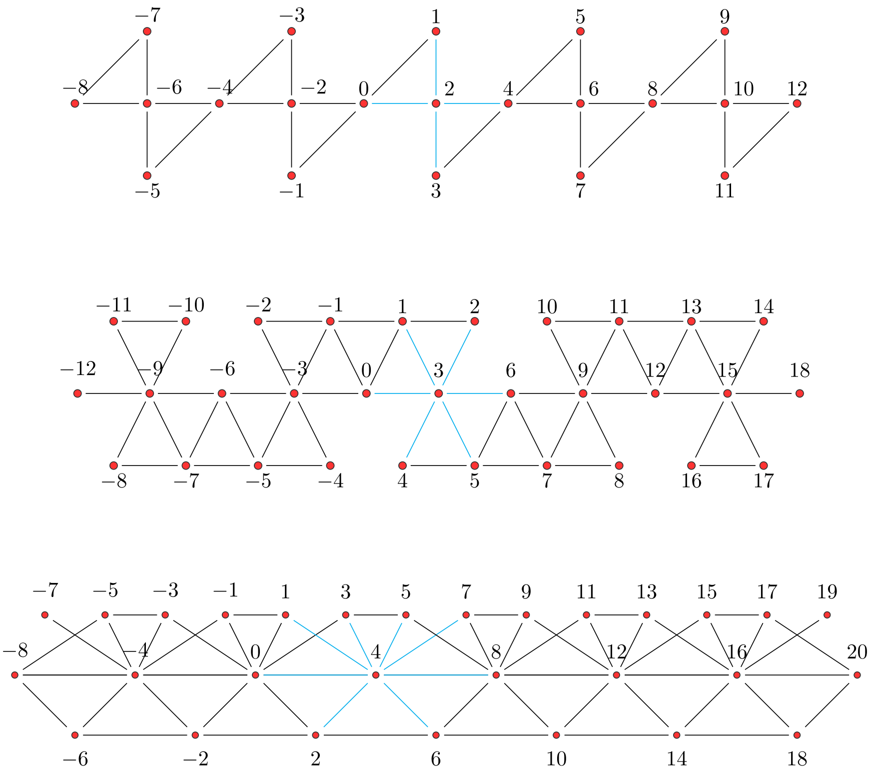

Fig. 5. Harmonic labelings obtained by completing a coalescence of infinite copies of $K_{1,4}$ (top), $K_{1,6}$ (middle) and $K_{1,8}$ (bottom), where the edges of the original weakly labeled graphs are shown in cyan. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

3.2. Labelings inferred from finite models

The weakly labeled graph in Fig. 3 (bottom) is a particular case of the family portrayed in Fig. 7, which we call $C^{k,h}$. We note that this collection can too be extended to $[0,\infty)\cap\mathbb{Z}$ and $\mathbb{Z}$, and that this last extension produces an harmonically labeled graph, $C^{k,\infty}$. Formally, $V_{C^{k,\infty}}=\mathbb{Z}$ and $E_{C^{k,\infty}}=\{\{a,b\} : b=a-1, a+1, a+k, a-k\}$. This new example of harmonic labeling is indeed part of a far more general family. Note that for $b\nsim a$ we can add the edges $(s+1)(b-a)+a\sim s(b-a)+a$ for each $s\in\mathbb{Z}$ and obtain a new harmonically labeled graph (see Fig. 8). We can repeat this process to the newly generated example to obtain infinitely many new ones (a different for each edge selected for addition and each k). We make this construction precise next.

Let $\mathcal{B}=\{(i,k) : k>1 \text{ and } 0\le i\le k-1\}$. For any (finite or infinite) subset B of $\mathcal{B}$ we form the graph P_B obtained from (the harmonically labeled graph) $\mathbb{Z}$ by adding the edges $\{(s+1)k+i, sk+i\}$ for every $s\in\mathbb{Z}$ for each $(i,k)\in B$. We call B a *base* for P_B and we write $P_B=\langle x : x\in B\rangle$ (the elements of B are the *spanning edges* of P_B). We picture a concrete example in Fig. 9.

Proposition 3.4. *For any $B\subset\mathcal{B}$, P_B is an harmonically labeled graph. Furthermore, $P_B=P_{B'}$ if and only if $B=B'$.*

Proof. First of all, we note that the set of edges added by different pairs (i,k) and (i',k') are disjoint. Indeed, the system

$$\begin{cases} sk+i=s'k'+i' \\ (s+1)k+i=(s'+1)k'+i' \end{cases}$$

has unique solution $s=s'$, $k=k'$ and $i=i'$ for $0\le i,i'\le k-1$. So it suffices to show that if a vertex v is harmonically labeled then adding the edges $\{(s+1)k+i, sk+i\}$ to a $P_{B'}$ corresponding to a single member $(i,k)\in B\setminus B'$ keeps v harmonic. This is clear if the vertex v is not incident to any of the added edges. Otherwise, v has new adjacent vertices labeled $\ell(v)-k$ and $\ell(v)+k$. Therefore

$$\sum_{w\sim v\in P_{B'}}\ell(w)+(\ell(v)-k)+(\ell(v)+k)=(\deg(v)+2)\ell(v),$$

which proves the claim. Finally, by the previous remarks, every edge is exclusive of a given (i,k) with $k\ge 2$ and $0\le i\le k-1$. Therefore, $P_B=P_{B'}$ if and only if $B=B'$. $\square$

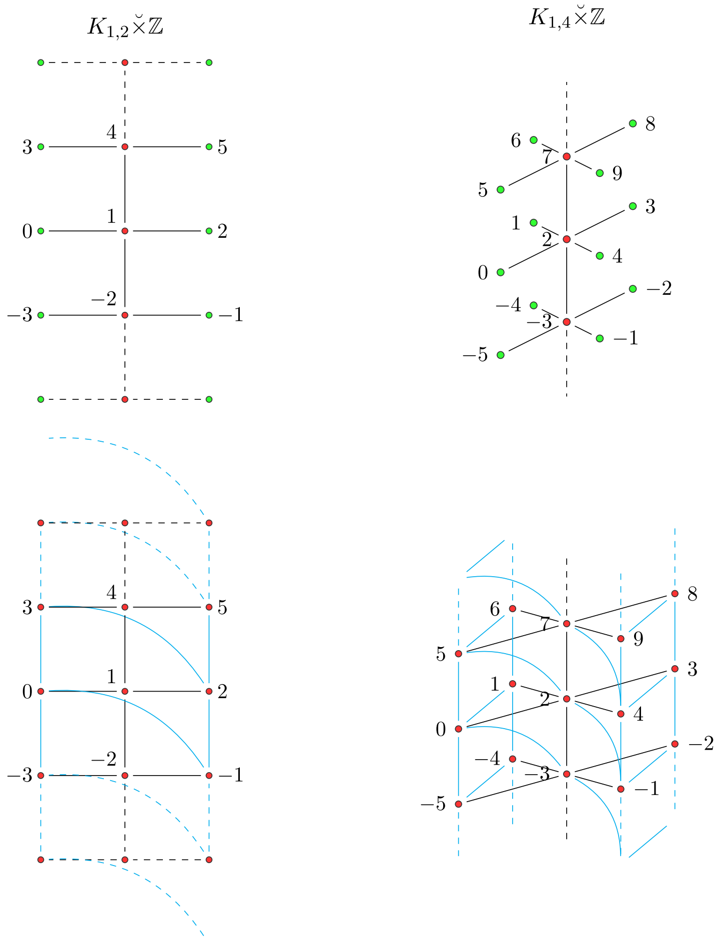

Fig. 6. Top. The weak harmonic labeling induced in the inner cylinder of $K_{1,2}$ (left) and $K_{1,4}$ (right). **Bottom.** Harmonic labeling from the weak labeling of $K_{1,2}\check{\times}\mathbb{Z}$ (left) and $K_{1,4}\check{\times}\mathbb{Z}$ (right). The cyan colored edges represent added edges to the original weak labelings.

Corollary 3.5. *The collection $\mathcal{P} = \{P_B : B \subset \mathcal{B}\}$ is a non-numerable family of harmonically labeled graphs.*

Some of the previously presented examples actually belong to the collection $\mathcal{P}$. For example, $C^{k,\infty} = \langle(0,k)\rangle$ and $K_{1,2}\check{\times}\mathbb{Z} = \langle(1,3)\rangle$. However, $K_{1,4}\check{\times}\mathbb{Z}$ is not one of these graphs.

Remark 3.6. A set $V' \subset V_G$ is said to be a *labeling spanning set* if the values of a labeling ℓ on the vertices of V' completely determine the labeling of G (by the harmonic property). In [1, §6, Open Problem 1] the authors ask which connected graphs other than $\mathbb{Z}$ admit an harmonic labeling spanned by a finite set. We claim that the members $\langle(0,k)\rangle$ of $P_{\mathcal{B}}$ for any $k \in \mathbb{Z}$ are finitely spanned by vertices labeled 0 and 1. Indeed, these two labels trivially determine all labels from 0 to k. The labels $x_{k+1}, x_{k+2}, \dots, x_{2k}$ pictured in Fig. 10 are solutions of the system

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 1 \\ 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_{k+1} \\ x_{k+2} \\ x_{k+3} \\ \vdots \\ x_{2k} \end{pmatrix} = \begin{pmatrix} 3k+1 \\ k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

whose matrix is non-singular for every $k \in \mathbb{Z}$. The claim is then settled by an inductive argument.

4. A characterization of weak harmonic labelings

In this section we characterize weakly labeled graphs in terms of certain collection of sets of integers which we call *harmonic subsets of* $\mathbb{Z}$.

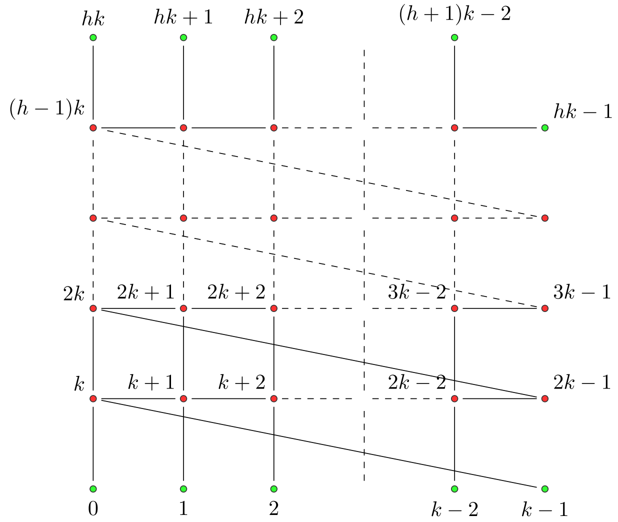

Fig. 7. The family $C^{k,h}$ of non-acyclic weakly labeled graphs which generalizes the family of Fig. 3 (bottom).

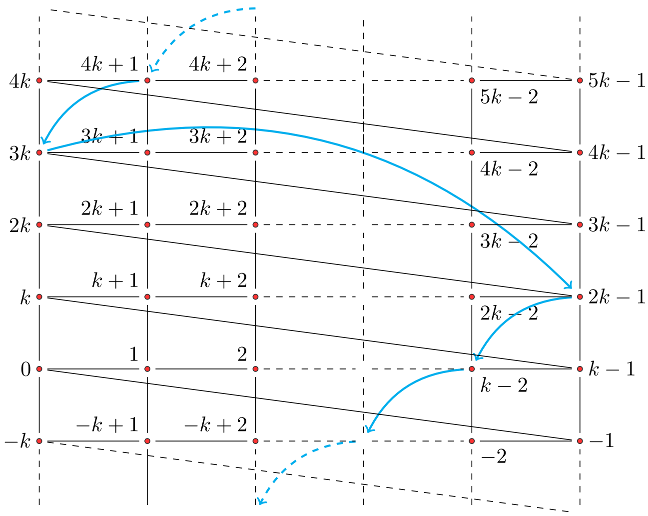

Fig. 8. New harmonic labeling from $C^{k,\infty}$.

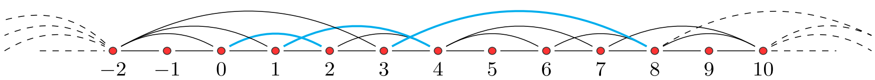

Fig. 9. The harmonically labeled graph $\langle(0,2),(1,3),(3,5)\rangle$ (in cyan, the spanning edges).

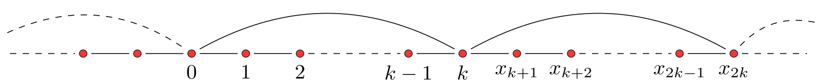

Fig. 10. The harmonic labeling of $\langle(0,k)\rangle$ is finitely spanned by $\{0,1\}$ for every $k \in \mathbb{Z}$.

Definition 4.1. Given a non-empty finite subset $A \subset \mathbb{Z}$ we let

$$av(A) = \frac{1}{|A|}\sum_{k\in A} k.$$

Here $|A|$ denotes the cardinality of A. We say that A is an *harmonic subset of* $\mathbb{Z}$ if $av(A) \in A$.

Remark 4.2. Note that every unit subset of $\mathbb{Z}$ is harmonic; we call them *trivial harmonic subsets*. Also, there are no two-element harmonic subsets of $\mathbb{Z}$. Therefore, any non-trivial harmonic subset of $\mathbb{Z}$ has at least three elements.

We shall show that certain collections of harmonic subsets of $\mathbb{Z}$ characterize weakly labeled graphs. For this, we consider pairs (G, ℓ) of a graph G and a weak harmonic labeling ℓ over G. Define an *isomorphism* between two weakly labeled graphs (G, ℓ) and (G', ℓ') as a graph isomorphism $f : G \to G'$ such that $\ell'(f(v)) = \ell(v)$ for every $v \in V_G$. We let $\mathcal{G}$ denote the quotient set of pairs (G, ℓ) under the isomorphism relation.

Given $(G, \ell) \in \mathcal{G}$ we consider the collection

$$\mathcal{A}_{(G,\ell)} = \{A_v : v \in V_G \setminus S_G\}$$

where $A_v = \{\ell(w) : w \in N_G(v)\}$. It is easy to see that $\mathcal{A}_{(G,\ell)}$ is a well-defined collection of non-trivial harmonic subsets of $\mathbb{Z}$ such that $av(A_v) = \ell(v)$. In particular, $A_v \neq A_u$ if $v \neq u$. Also, this collection is finite if and only if G is finite. Furthermore, the collection $\mathcal{A}_{(G,\ell)}$ satisfies the following conditions (whose easy verification are left to the reader).

Lemma 4.3. *Let $\mathcal{A}$ be the collection $\mathcal{A}_{(G,\ell)}$ of harmonic subsets of $\mathbb{Z}$ defined as above. For $A, B \in \mathcal{A}$, we have:*

(P1) $\bigcup_{C\in\mathcal{A}} C$ is an integer interval.
(P2) $av(A) \neq av(B)$ if $A \neq B$.
(P3) If $t \in A \cap B$, $A \neq B$, then there exists $C \in \mathcal{A}$ such that $av(C) = t$.
(P4) If $av(A) \in B$ then $av(B) \in A$.

Note that *(P2)* implies that the t in *(P3)* is unique. Actually, *(P2)* is covered by requesting the unicity of t in *(P3)*. However, we state it in this form for computational reasons that will become evident later.

The main result of this section is that properties *(P1)* through *(P4)* of Lemma 4.3 characterize weak harmonic labelings, in the sense that $(G, \ell) \mapsto \mathcal{A}_{(G,\ell)}$ is a bijection between $\mathcal{G}$ and the class $\mathcal{H}$ of collections of non-trivial harmonic subsets of $\mathbb{Z}$ satisfying *(P1)* through *(P4)*. Furthermore, if $\mathcal{G}_I \subset \mathcal{G}$ is the subset of pairs (G, ℓ) for which ℓ is a weak harmonic labeling onto I and $\mathcal{H}_I \subset \mathcal{H}$ is the class of collections $\mathcal{A}$ for which $\bigcup_{C\in\mathcal{A}} C = I$ then the bijection takes $\mathcal{G}_I$ onto $\mathcal{H}_I$.

Note that the map $(G, \ell) \mapsto \mathcal{A}_{(G,\ell)}$ sends elements of $\mathcal{G}_I$ to $\mathcal{H}_I$ by Remark 2.1. We next build the inverse map $\mathcal{H}_I \to \mathcal{G}_I$. Let $\mathcal{A} = \{A_i\}_{i\in J} \in \mathcal{H}_I$. We define the associated graph $G_{\mathcal{A}}$ as follows:

- $V_{G_{\mathcal{A}}} = I$
- $i \sim j \Leftrightarrow$ (there exists a t such that $i = av(A_t)$ and $j \in A_t$) or (there exists a t such that $j = av(A_t)$ and $i \in A_t$).

Furthermore, we define a vertex labeling $\ell_{\mathcal{A}} : V_{G_{\mathcal{A}}} \to I$ by $\ell_{\mathcal{A}}(i) = i$. Lemma 4.4 and Corollary 4.5 below prove that $(G_{\mathcal{A}}, \ell_{\mathcal{A}}) \in \mathcal{G}_I$.

Lemma 4.4. *With the notations as above, $i \in V_{G_{\mathcal{A}}} \setminus S_{G_{\mathcal{A}}}$ if and only if $\exists t \in J$ such that $i = av(A_t)$. Furthermore, this t is unique and $N_{G_{\mathcal{A}}}(i) = A_t$.*

Proof. If $i \in V_{G_{\mathcal{A}}} \setminus S_{G_{\mathcal{A}}}$ then there exist $j_1 \neq j_2$ such that $j_1, j_2 \in N_{G_{\mathcal{A}}}(i)$. If $i \neq av(A_t)$ for every t then $\exists t_1, t_2$ such that $j_1 = av(A_{t_1})$, $j_2 = av(A_{t_2})$ and $i \in A_{t_1} \cap A_{t_2}$. But then *(P3)* implies the existence of t such that $i = av(A_t)$, contradicting our assumption.

Suppose now that $i = av(A_t)$ for some t. In particular, $i \in A_t$. By Remark 4.2, there exist $j_1, j_2 \in A_t$ non-equal such that $j_1, j_2 \neq av(A_t)$. Thus $j_1, j_2 \in N_{G_{\mathcal{A}}}(i)$ by the definition of adjacency in $G_{\mathcal{A}}$ and hence $i \in V_{G_{\mathcal{A}}} \setminus S_{G_{\mathcal{A}}}$.

The uniqueness of t is a direct consequence of *(P2)*. Now, if $j \in N_{G_{\mathcal{A}}}(i)$ then either (there exists s such that $i = av(A_s)$ and $j \in A_s$) or (there exists s such that $j = av(A_s)$ and $i \in A_s$). In the first case $s = t$ by unicity. In the latter, *(P4)* implies that $j = av(A_s) \in A_t$. In any case $j \in A_t$, which proves $N_{G_{\mathcal{A}}}(i) \subset A_t$. Now, if $j \in A_t$ then $i \sim j$ by the definition of adjacency of $G_{\mathcal{A}}$. Hence, $j \in N_{G_{\mathcal{A}}}(i)$. □

Corollary 4.5. *With the notations as above, $\ell_{\mathcal{A}}$ is a weak harmonic labeling over $G_{\mathcal{A}}$.*

Proof. If $i \in V_{\mathcal{A}} \setminus S_{\mathcal{A}}$, let t be such that $i = av(A_t)$. Then

$$\ell_{\mathcal{A}}(i) = i = av(A_t) = \frac{1}{|A_t|}\sum_{k\in A_t} k = \frac{1}{|N_{G_{\mathcal{A}}}(i)|}\sum_{k\in N_{G_{\mathcal{A}}}(i)} k = \frac{1}{\deg(i)+1}\sum_{\substack{k\sim i\\ k=i}} \ell_{\mathcal{A}}(k). \quad \square$$

Theorem 4.6. *The maps $(G, \ell) \mapsto \mathcal{A}_{(G,\ell)}$ and $\mathcal{A} \mapsto (G_{\mathcal{A}}, \ell_{\mathcal{A}})$ are mutually inverse.*

Proof. Define the function $f:(G,\ell)\to(G_{\mathcal{A}_{(G,\ell)}},\ell_{\mathcal{A}_{(G,\ell)}})$ as $f(v)=\ell(v)$. We will show that f is a graph isomorphism between G and $G_{\mathcal{A}_{(G,\ell)}}$ and that $\ell_{\mathcal{A}_{(G,\ell)}}(f(v))=\ell(v)$. Since ℓ is a weak harmonic labeling then f is a bijection between V_G and I, so it suffices to show that $v\sim w$ if and only if $f(v)\sim f(w)$. Now, if $v\sim w$ then either v or w must belong to the set of non-leaves of G (Remark 2.1). Assume $v\in V_G\setminus S_G$. Then, by definition of $\mathcal{A}_{(G,\ell)}$ it exists A_v with $av(A_v)=\ell(v)$. Also, since $v\sim w$ then $w\in N_G(v)$ and hence $\ell(w)\in A_v$. Therefore $\ell(v)\sim\ell(w)$; that is, $f(v)\sim f(w)$.

Now, suppose $f(v)\sim f(w)$. Then $\ell(v)\sim\ell(w)$ in $G_{\mathcal{A}_{(G,\ell)}}$. Then, either (there exists $u\in V_G\setminus S_G$ such that $\ell(v)=av(A_u)$ and $\ell(w)\in A_u$) or (there exists $x\in V_G\setminus S_G$ such that $\ell(w)=av(A_x)$ and $\ell(v)\in A_x$). Without loss of generality we may assume the first case happens. Since ℓ is a bijection then w must belong to $N_G(v)$. Hence $w\sim v$. This proves that G is isomorphic to $G_{\mathcal{A}_{(G,\ell)}}$.

Finally, from the definition of $\ell_{\mathcal{A}_{(G,\ell)}}$:

$$\ell_{\mathcal{A}_{(G,\ell)}}(f(v))=f(v)=\ell(v),$$

which concludes the proof that $(G,\ell)\mapsto\mathcal{A}_{(G,\ell)}\mapsto(G_{\mathcal{A}_{(G,\ell)}},\ell_{\mathcal{A}_{(G,\ell)}})$ is the identity.

We now prove that $\mathcal{A}\mapsto(G_{\mathcal{A}},\ell_{\mathcal{A}})\mapsto\mathcal{A}_{(G_{\mathcal{A}},\ell_{\mathcal{A}})}$ is the identity. Define $g:\mathcal{A}\to\mathcal{A}_{(G_{\mathcal{A}},\ell_{\mathcal{A}})}$ as follows: $g(A_t)=\tilde{A}_i$ where $i\in V_{G_{\mathcal{A}}}\setminus S_{G_{\mathcal{A}}}$ is such that $i=av(A_t)$ (Lemma 4.4). Note that g is one to one by (*P2*) and the fact that $i\neq j$ implies $\tilde{A}_i\neq\tilde{A}_j$ in $\mathcal{A}_{(G_{\mathcal{A}},\ell_{\mathcal{A}})}$ (see properties of $\mathcal{A}_{(G,\ell)}$ before Lemma 4.3). Also, Lemma 4.4 implies that g is onto. Since $\ell_{\mathcal{A}}(s)=s$ and $A_t=N_{G_{\mathcal{A}}}(i)$ (again by Lemma 4.4) then $\tilde{A}_i=\{\ell_{\mathcal{A}}(s):s\in N_{G_{\mathcal{A}}}(i)\}=N_{G_{\mathcal{A}}}(i)=A_t$. □

We shall exhibit evidence in the next section that Theorem 4.6 provides a more efficient way to compute weak harmonic labelings of finite graphs than an exhaustive search calculation.

If we now let $\tilde{\mathcal{G}}_{\mathbb{Z}}\subset\mathcal{G}_{\mathbb{Z}}$ denote the set of pairs (G,ℓ) for which $S_G=\emptyset$ then $\tilde{\mathcal{G}}_{\mathbb{Z}}$ is the set of harmonically labeled graphs as defined in [1]. From Theorem 4.6 we obtain the following characterization of harmonic labelings.

Corollary 4.7. *Let G be a graph and $\ell:V_G\to\mathbb{Z}$. Then ℓ is an harmonic labeling of G if and only if $S_G=\emptyset$ and $\mathcal{A}_{(G,\ell)}$ satisfies:*

(ZP1) For every $k\in\mathbb{Z}$ there exists $C\in\mathcal{A}_{(G,\ell)}$ such that $av(C)=k$.
(ZP2) $av(A)\neq av(B)$ if $A\neq B$.
(ZP3) If $av(A)\in B$ then $av(B)\in A$.

Proof. The result follows from Theorem 4.6 by noting that *P1* transforms into *ZP1* and that *P3* is covered by *ZP1*. □

Remark 4.8. Weakly labeled *connected* graphs can also be entirely characterized in terms of harmonic subsets of $\mathbb{Z}$. It suffices to add the following property to those of Lemma 4.3:

(P5) There exists a sequence $A_{i_1},\dots,A_{i_r}\subset\mathcal{A}$ such that $A_{i_1}=A$, $A_{i_r}=B$ and $av(A_{i_j})\in A_{i_{j+1}}$ for $1\leq j\leq r-1$.

It is straightforward to see that (G,ℓ) (resp. $G_{\mathcal{A}}$) is connected if and only if $\mathcal{A}_{(G,\ell)}$ (resp. $\mathcal{A}$) satisfies *(P5)*.

5. On computing weakly labeled finite graphs

The characterization given in Theorem 4.6 provides a much less costly method to compute all weakly labeled graphs of $n+1$ vertices than testing the finite number of all possible graphs and all possible bijective maps from V_G to $[0,n]\cap\mathbb{Z}$. We shall provide evidence to this assertion in the present section.

Let H_n represent the set of non-trivial harmonic subsets of $[0,n]\cap\mathbb{Z}$. Property *(P2)* in Lemma 4.3 grants a restriction which can be used to significantly reduce the collections of harmonic subsets of $[0,n]\cap\mathbb{Z}$ to be considered (in comparison to the powerset of H_n). Indeed, since the averages of the sets cannot be repeated one may form collections by picking at most one set for each possible average. If we let h_k denote the number of non-trivial harmonic subsets of $[0,n]\cap\mathbb{Z}$ of average k ($1\leq k\leq n-1$) then the number of possible (non-trivial) collections of this type is

$$\gamma_n=\prod_{k=1}^{n-1}(h_k+1)-1.$$

Let Γ_n represent the set of collections of harmonic subsets of $[0,n]\cap\mathbb{Z}$ satisfying *(P2)* (so $\gamma_n=|\Gamma_n|$). Table 1 shows the number of non-trivial harmonic subsets of $[0,n]\cap\mathbb{Z}$ grouped by average (needed to compute γ_n) and Table 2 (left) shows the ratio between γ_n and the number $2^{\binom{n+1}{2}}$ of possible graphs of $n+1$ vertices bijectively labeled with $[0,n]\cap\mathbb{Z}$ for the first positive integers. The latter evidences that the size of the input data to be analyzed for an harmonicity condition is vastly smaller if we use the characterization provided in Theorem 4.6.

Now, it is not hard to see that the time cost of computing H_n with its elements sorted according to their average is under $(n+1)^2 2^{n+1}$, where we assume comparisons, assignments, variable reading and basic operations have a constant run time

Table 1
Number of non-trivial harmonic subsets of $[0, n] \cap \mathbb{Z}$ disaggregated by average: for each n, the ith entry in the $(n-1)$-tuple on the second column is the number of non-trivial harmonic subsets of $[0, n] \cap \mathbb{Z}$ with average i.

n	Average distribution	γ_n
3	[1, 1]	3
4	[1, 3, 1]	15
5	[1, 4, 4, 1]	99
6	[1, 4, 9, 4, 1]	999
7	[1, 4, 12, 12, 4, 1]	16899
8	[1, 4, 14, 25, 14, 4, 1]	584999
9	[1, 4, 15, 37, 37, 15, 4, 1]	36966399
10	[1, 4, 15, 46, 75, 46, 15, 4, 1]	4297830399
11	[1, 4, 15, 52, 117, 117, 52, 15, 4, 1]	1001280409599
12	[1, 4, 15, 55, 154, 235, 154, 55, 15, 4, 1]	455188643839999
13	[1, 4, 15, 57, 183, 379, 379, 183, 57, 15, 4, 1]	421016185077759999
14	[1, 4, 15, 58, 204, 525, 759, 525, 204, 58, 15, 4, 1]	787475029337190399999
15	[1, 4, 15, 58, 218, 654, 1260, 1260, 654, 218, 58, 15, 4, 1]	$\approx 2.91571446499836 \times 10^{24}$
16	[1, 4, 15, 58, 227, 758, 1814, 2521, 1814, 758, 227, 58, 15, 4, 1]	$\approx 2.21715254911351 \times 10^{28}$
17	[1, 4, 15, 58, 233, 839, 2353, 4277, 4277, 2353, 839, 233, 58, 15, 4, 1]	$\approx 3.49163431528466 \times 10^{32}$

Table 2
Left. Ratio between the number of collections of non-trivial harmonic subsets of $[0, n] \cap \mathbb{Z}$ fulfilling property $(P2)$ and the number of graphs on $n+1$ vertices bijectively labeled with $\{0, 1, \ldots, n\}$. **Middle.** Ratio between the time cost of computing γ_n and the estimated time cost of computing all harmonic subsets of $[0, n] \cap \mathbb{Z}$ sorted by average. **Right.** Ratio between the estimated time cost of computing Γ_n and checking Properties *(P1)*, *(P3)* and *(P4)* versus the time cost of checking for weak harmonicity in every possible graph of $n+1$ vertices bijectively labeled with $[0, n] \cap \mathbb{Z}$.

n	$\gamma_n / 2^{\binom{n+1}{2}}$	$\gamma_n / ((n+1)^2 2^{n+1})$	$((n+1)^5 \gamma_n) / (n 2^{\binom{n+1}{2}})$
3	0.046875	0.01171875	21.3333333333
4	0.0146484375	0.01875	12.20703125
5	0.00302124023438	0.04296875	4.74609375
6	0.000476360321045	0.159279336735	1.33570035299
7	$6.29536807537 \cdot 10^{-5}$	1.03143310547	0.294712611607
8	$8.51285585668 \cdot 10^{-6}$	14.1058786651	0.0628345605946
9	$1.05064825107 \cdot 10^{-6}$	360.999990234	0.0116738697721
10	$1.19288756623 \cdot 10^{-7}$	17343.3884257	0.00192115735473
11	$1.35698799419 \cdot 10^{-8}$	1697591.84028	0.000306965487792
12	$1.50609288496 \cdot 10^{-9}$	328787100.592	$4.66001454611 \cdot 10^{-5}$
13	$1.70047082943 \cdot 10^{-10}$	$1.3110606449 \cdot 10^{11}$	$7.03503094898 \cdot 10^{-6}$
14	$1.94127582651 \cdot 10^{-11}$	$1.06808136575 \cdot 10^{14}$	$1.05296880768 \cdot 10^{-6}$
15	$2.19353976462 \cdot 10^{-12}$	$1.73790124953 \cdot 10^{17}$	$1.53339543482 \cdot 10^{-7}$
16	$2.54516630507 \cdot 10^{-13}$	$5.85312509798 \cdot 10^{20}$	$2.25860762151 \cdot 10^{-8}$
17	$3.0580136275 \cdot 10^{-14}$	$4.11096539354 \cdot 10^{24}$	$3.39901452593 \cdot 10^{-9}$

under 1 (see Algorithm 1 in Table 3). Table 2 (middle) shows the ratio between this number and γ_n for the first positive integers, which evidences that the time complexity of computing Γ_n is $O(\gamma_n)$. By Table 2 (left) again, this time is better than the time complexity $2^{\binom{n+1}{2}}$ of computing every possible graph of $n+1$ vertices bijectively labeled with $[0, n] \cap \mathbb{Z}$. As to the verification of Properties *(P1)*, *(P3)* and *(P4)*, the algorithms shown in Table 3 evidence that the time cost for checking all three properties is loosely bounded above by $(n+1)^5$. Finally, Table 2 (right) shows the ratio between the estimated time cost of computing Γ_n and checking Properties *(P1)*, *(P3)* and *(P4)* versus the time cost of computing every possible graph of $n+1$ vertices bijectively labeled with $[0, n] \cap \mathbb{Z}$ and verifying weak harmonicity, where we have assumed a linear time complexity for this last task.

In Table 4 we present all possible weakly labeled graphs of up to (and including) 10 vertices computed using the characterization given in Theorem 4.6.

6. Multigraphs and total labelings

In this section we extend the main definitions and results of weak harmonic labelings to multigraphs and provide a generalization of Theorem 4.6 in this context. *All multigraphs have connected components of at least 3 vertices, are loopless and have bounded degree (see Remark 6.7).* Also, since the identity of the edges is indifferent to the theory, we consider all parallel edges to be indistinguishable.

Recall that a (finite) *multiset* $\mathcal{M}$ is a pair (A, m) where A is a (finite) non-empty set and $m : A \to \mathbb{N}$ is a function giving the *multiplicity* of each element in A (the number of instances of that element). The cardinality of $\mathcal{M}$ is the number $|\mathcal{M}| = \sum_{x \in A} m(x)$. If $A = \{x_1, x_2, \cdots x_n\}$ we shall often write $\mathcal{M} = \{x_1^{m(x_1)}, x_2^{m(x_2)}, \ldots, x_n^{m(x_n)}\}$. If $m(x_i) = 1$ we simply write x_i.

Table 3
Algorithms for computing the collection of all harmonic subsets of $[0,n]\cap\mathbb{Z}$ sorted by average and Properties *(P1)*, *(P3)* and *(P4)*. These algorithms are deliberately based on detailed non-optimal procedures to strengthen the evidence of the efficiency of computing by means of the characterization given in Theorem 4.6 versus the exhaustive calculation.

Algorithm 1 Computing non-trivial harmonic subsets of $[0,n]\cap\mathbb{Z}$ sorted by average.

```
Require: n ≥ 2
Ensure: H_n sorted by average
for i ∈ [1, n − 1] ∩ Z do
    H_n^i ← {}
end for
for B ∈ Powerset({0, ..., n}) do
    compute av(B)
    if |B| ≥ 3 and av(B) ∈ B then
        H_n^{av(B)} ← H_n^{av(B)} ∪ {B}
    end if
end for
return H_n = (H_n^1, H_n^2, ..., H_n^{n−1})
```

Algorithm 2 Verifying Property (P1).

```
Require: A collection C of harmonic subsets of [0, n] ∩ Z of different
    averages
Ensure: True iff C satisfies Property (P1)
for i ∈ [0, n] ∩ Z do
    check_i ← 0
    for A ∈ C do
        if i ∈ A then
            check_i ← 1
        end if
    end for
end for
return check_i = 1 for all i ∈ [0, n] ∩ Z
```

Algorithm 3 Verifying Property (P3).

```
Require: A collection C of harmonic subsets of [0, n] ∩ Z of different
    averages
Ensure: True iff C satisfies Property (P3)
T ← {}
for A, B ∈ C do
    compute A ∩ B
    for t ∈ A ∩ B do
        T ← T ∪ {t}
    end for
end for
averages ← {}
for A ∈ C do
    compute av(A)
    averages ← averages ∪ {av(A)}
end for
check ← 1
for t ∈ T do
    if t ∉ averages then
        check ← 0
    end if
end for
return check = 1
```

Algorithm 4 Verifying Property (P4).

```
Require: A collection C of harmonic subsets of [0, n] ∩ Z of different
    averages
Ensure: True iff C satisfies Property (P4)
averages ← {}
for A ∈ C do
    compute av(A)
    averages ← averages ∪ {av(A)}
end for
check ← 1
for A ∈ C do
    for t ∈ A, t ≠ av(A) do
        if t ∈ averages then
            check_t ← 0
            for B ∈ C, B ≠ A do
                if t ∈ B then
                    check_t ← 1
                end if
            end for
        end if
    end for
end for
return check_t ≠ 0 for all t ∈ averages
```

Given a multigraph G we let $m_G(v,w)=m_G(w,v)\in\mathbb{Z}_{\geq 0}$ denote the number of edges between vertices $v,w\in V_G$, $v\neq w$. If $m_G(v,w)\neq 0$ then v and w are adjacent and we write $v\sim w$. If $m(v,w)=k\geq 2$ we shall often write $v\overset{k}{\sim}w$ or $\{v,w\}^k\in G$. A vertex $v\in G$ is a *leaf* if $m_G(v,w)\neq 0$ for exactly one $w\neq v$. As in the simple case, we shall denote S_G the set of leaves of the multigraph G. The *simplification* of a multigraph G is the simple graph sG where $V_{sG}=V_G$ and $\{u,v\}\in E_{sG}$ if and only if $m_G(v,w)\neq 0$ ($v\neq w$). We shall call the *closed multi neighborhood* of $v\in V_G$ in a multigraph G to the multiset $\mathcal{N}_G(v)=\{v\}\cup\{w^{m_G(v,w)}: v\sim w\}$. Thus, the closed multi neighborhood of v keeps track of the multiplicities of the vertices adjacent to v as well. The (standard) close neighborhood of v is $N_{sG}(v)\subset V_G$.

Definition 6.1. A *weak harmonic labeling* of a multigraph G is a bijective function $\ell: V_G\to I$ such that

$$\ell(v)=\frac{1}{deg(v)}\sum_{w\sim v} m_G(v,w)\phi(w) \qquad \forall v\in V_G\setminus S_G.$$

Fig. 11 shows some examples of harmonic labelings of finite multigraphs. Note that the presence of at least two leaves is still a requirement for the existence of a weak harmonic labeling.

We next show that Theorem 4.6 can be generalized to multigraphs.

Definition 6.2. For a multiset $\mathcal{M}=(A,m)$ with finite non-empty $A\subset\mathbb{Z}$ we let

$$av(\mathcal{M})=\frac{1}{|\mathcal{M}|}\sum_{k\in A} m(k)k.$$

Table 4
All weakly labeled graphs of up to (and including) 10 vertices. For simplicity, harmonic subsets have been written in the form $a_1a_2\cdots a_s$ instead of $\{a_1, a_2, \ldots, a_s\}$. For example, 23406 denotes the harmonic subset {2, 3, 4, 0, 6}.

n	Harmonic Subsets of $[0, n]$
2	{012}
3	{012, 321}
4	{012, 321, 432}; {12304}
5	{012, 321, 432, 543}; {012, 543}; {204, 351}
6	{012, 321, 432, 543, 654}; {012, 321, 654}; {012, 543, 654}; {204, 351, 462}; {12304, 432, 54362}; {321, 23406, 543}; {1234506}
7	{012, 321, 432, 543, 654, 765}; {012, 321, 432, 765}; {012, 321, 654, 765}; {012, 543, 654, 765}; {012, 65473}; {204, 351, 462, 573}; {204, 6543271}; {12304, 765}; {321, 2307, 654}; {321, 4507, 654}; {2307, 5461}; {13407, 54362}; {3261, 4507}; {23406, 54371}; {1234506, 573}; {43251, 34607}
8	{012, 321, 432, 543, 654, 765, 876}; {012, 321, 432, 543, 876}; {012, 321, 432, 765, 876}; {012, 321, 654, 765, 876}; {012, 321, 76584}; {012, 543, 654, 765, 876}; {012, 543, 876}; {012, 573, 684}; {1205, 543, 65482, 765}; {1205, 7654382}; {204, 351, 462, 573, 684}; {204, 351, 876}; {204, 54281, 65473}; {204, 6543271, 684}; {204, 4372, 6581}; {204, 54362, 75481}; {204, 765381}; {12304, 432, 54362, 654, 76584}; {12304, 462, 76584}; {12304, 765, 876}; {321, 23406, 543, 7683}; {321, 432, 34508, 654, 765}; {321, 432, 543, 456708}; {321, 408, 765}; {321, 456708}; {123408, 543, 654, 765}; {123408, 765}; {2307, 5461, 684}; {13407, 54362, 684}; {13407, 543, 65482}; {123507, 684}; {306, 471, 582}; {23406, 543, 75481}; {1234506, 7683}; {351, 24608, 573}; {351, 245607, 5483}; {43251, 34508, 65473}; {43251, 34607, 684}; {3405, 643281, 573}; {3405, 6543271, 5483}; {432, 1345708, 654}; {123456708}
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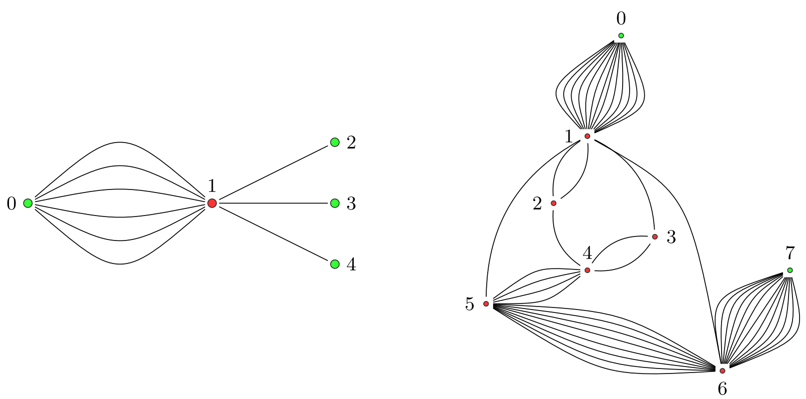

Fig. 11. Examples of weak harmonic labeling on multigraphs.

We say that $\mathcal{M}$ is an *harmonic multiset of* $\mathbb{Z}$ if $av(\mathcal{M}) \in A$.

Remark 6.3. As for harmonic subsets, the multisets whose underlying set is a unit set of $\mathbb{Z}$ are (trivial) harmonic multisets of $\mathbb{Z}$. Also, there are no harmonic multisets of $\mathbb{Z}$ whose underlying set has two elements. Therefore, any non-trivial harmonic multiset of $\mathbb{Z}$ has an underlying set of at least three elements.

Analogously to the simple case, we consider pairs (G,ℓ) for a multigraph G and a weak harmonic labeling $\ell : V_G \to I$ and define an *isomorphism* between two weakly labeled multigraphs (G,ℓ) and (G',ℓ') as a multigraph isomorphism $f : G \to G'$ such that $\ell(f(v)) = \ell(v)$ for every $v \in V_G$. We let $\mathcal{MG}_I$ denote the quotient set of pairs (G,ℓ), $\ell : V_G \to I$, under the isomorphism relation.

Given $(G,\ell) \in \mathcal{MG}_I$ we consider the collection

$$\mathcal{MA}_{(G,\ell)} = \{\mathcal{B}_v \,:\, v \in V_G \setminus S_G\}$$

where $\mathcal{B}_v = \{\ell(v)\} \cup \{\ell(w)^{m_G(v,w)} \,:\, w \sim v\}$. As in the simple case, it is easy to see that $\mathcal{MA}_{(G,\ell)}$ is a collection of non-trivial harmonic multisets of $\mathbb{Z}$ verifying $av(\mathcal{B}_v) = \ell(v)$ that satisfies the (analogous) conditions than Lemma 4.3. Namely, if $A_{\mathcal{M}}$ stands for the underlying set of the multiset $\mathcal{M}$:

Lemma 6.4. *Let $\mathcal{MA}$ be the collection $\mathcal{MA}_{(G,\ell)}$ of harmonic multisets of $\mathbb{Z}$ defined as above. For $\mathcal{B}, \mathcal{C} \in \mathcal{MA}$, we have:*

(MP1) $\bigcup_{\mathcal{D} \in \mathcal{MA}} A_{\mathcal{D}} = I$.
(MP2) $av(\mathcal{B}) \neq av(\mathcal{C})$ *if* $\mathcal{B} \neq \mathcal{C}$.
(MP3) *If* $t \in A_{\mathcal{B}} \cap A_{\mathcal{C}}$ *then there exists* $\mathcal{D} \in \mathcal{MA}$ *such that* $av(\mathcal{D}) = t$.
(MP4) *If* $av(\mathcal{B})^k \in \mathcal{C}$ *then* $av(\mathcal{C})^k \in \mathcal{B}$.

We let $\mathcal{MH}_I$ stand for the class of collections of non-trivial harmonic multisets of $\mathbb{Z}$ with $\bigcup_{\mathcal{D} \in \mathcal{MA}} A_{\mathcal{D}} = I$ satisfying *(MP1)* through *(MP5)* of Lemma 6.4. With the analogous constructions as in the simple case it can be shown that there is a bijection $\mathcal{MG}_I \equiv \mathcal{MH}_I$. Namely, for $\mathcal{MA} = \{\mathcal{B}_i\}_{i \in J} \in \mathcal{MH}_I$ define the associated multigraph $G_{\mathcal{MA}}$ as:

- $V_{\mathcal{MA}} = I$
- $i \overset{k}{\sim} j \in G_{\mathcal{MA}} \Leftrightarrow (\exists t / i = av(\mathcal{B}_t)$ and $j^k \in \mathcal{B}_t)$ or $(\exists t / j = av(\mathcal{B}_t)$ and $i^k \in \mathcal{B}_t)$

Note that, by *(MP4)*, this multigraph is well-defined. Finally, we define a vertex labeling $\ell_{\mathcal{MA}}$ over $G_{\mathcal{MA}}$ by $\ell_{\mathcal{MA}}(i) = i$.

Identical arguments as in the proofs of Lemma 4.4, Corollary 4.5 and Theorem 4.6 go through to prove the following analogous results for multigraphs.

Lemma 6.5. *With the notations as above,*

(1) $G_{\mathcal{MA}}$ *is connected.*
(2) $i \in V_{G_{\mathcal{MA}}} \setminus S_{G_{\mathcal{MA}}}$ *if and only if* $\exists t \in J$ *such that* $i = av(\mathcal{B}_t)$. *Furthermore, this* t *is unique and* $\mathcal{N}_{G_{\mathcal{MA}}}(i) = \mathcal{B}_t$. *In particular,* $j \in A_{\mathcal{B}_t}$ *if and only if* $j = i$ *or* $j \sim i$ *in* $G_{\mathcal{MA}}$.
(3) $\ell_{\mathcal{MA}}$ *is a weak harmonic labeling over* $G_{\mathcal{MA}}$.

Theorem 6.6. *The maps* $(G,\ell) \to \mathcal{MA}_{(G,\ell)}$ *and* $\mathcal{MA} \to (G_{\mathcal{MA}}, \ell_{\mathcal{MA}})$ *are mutually inverse.*

Remark 6.7. All the results of this section can be extended in a straightforward manner to multigraphs with loops. This is consequence of the fact that a multiset

$$\{x_1^{m_1}, x_2^{m_2}, \ldots, x_k, \ldots, x_n^{m_n}\}$$

is harmonic with average x_k if and only if $\{x_1^{m_1}, x_2^{m_2}, \ldots, x_k^{m}, \ldots, x_n^{m_n}\}$ is harmonic with average x_k for all $m > 0$.

6.1. Total weak harmonic labelings

Since a weak harmonic labeling over a multigraph G is trivially equivalent to a total labeling over sG we can state the theory in terms of total labelings.

Definition 6.8. If G is a simple graph, then we call a *total weak harmonic labeling* of G onto I to a function $\ell : V_G \cup E_G \to \mathbb{Z}$ such that $\ell|_{V_G}$ is a bijection with I and

Algorithm 5 Total weak harmonic labeling.

Require: $\phi : V_G \to [0, n-1] \cap \mathbb{Z}$ with property (3)
Ensure: ℓ a total harmonic labeling on G
procedure TOTALLABELINGFROM(ϕ)
  Order $V_G \setminus S_G = \{v_1, \ldots, v_t\}$ such that $\phi(v_i) < \phi(v_j)$ if $i < j$.
  $\mathcal{B}_i \leftarrow \{\phi(w) \mid w \in N_G(v_i)\}$ $(1 \le i \le t)$.
  for $1 \le i \le t$ **do**
    if $av(\mathcal{B}_i) \neq \phi(v_i)$ **then**
      Harmonize $\mathcal{B}_i$ by conveniently altering the multiplicity of the elements different from $\phi(v_i)$ (see Remark 6.9).
      if $\phi(v_j) \in \mathcal{B}_i$ **then**
        For $1 \le j < i$**:** Correct the multiplicities of the elements of $\mathcal{B}_j$ so the multiplicity of $\phi(v_i) \in \mathcal{B}_j$ coincides with that of $\phi(v_j) \in \mathcal{B}_i$
        For $i < j \le t$**:** Correct the multiplicity of $\phi(v_i) \in \mathcal{B}_j$ so it coincides with that of $\phi(v_j) \in \mathcal{B}_i$
      end if
    end if
  end for
  $\ell(v) \leftarrow \phi(v)$ for every $v \in V_G$
  $\ell(\{w, u\}) \leftarrow$ multiplicity of $\phi(u)$ in $\mathcal{B}_{\phi(w)}$
end procedure

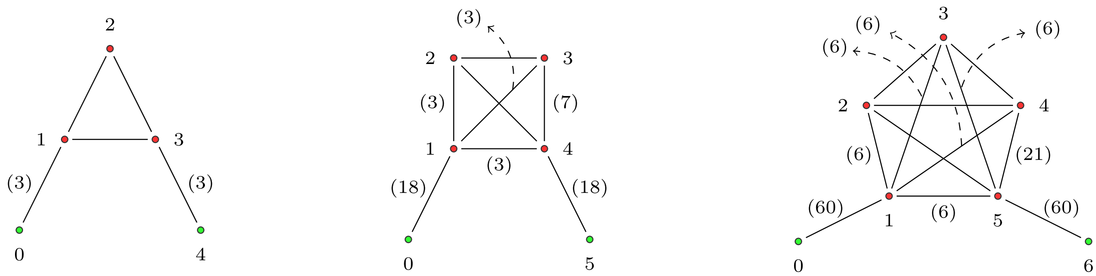

Fig. 12. Examples of total weak harmonic labelings obtained from Algorithm 5. The labels of the edges appear in parentheses (labels equal to 1 are omitted).

$$\ell(v) = \frac{1}{deg(v)} \sum_{w \sim v} \ell(\{v, w\})\ell(w) \qquad \forall v \in V_G \setminus S_G.$$

Given a weak harmonic labeling $\ell : V \to I$ over a multigraph G we have the associated total weak harmonic labeling $\ell^* : V_{sG} \cup E_{sG} \to \mathbb{Z}$ over sG defined as

$$\begin{cases} \ell^*(v) = \ell(v) & v \in V_{sG} \\ \ell^*(\{v, w\}) = m_G(v, w) & \{u, v\} \in E_{sG}. \end{cases}$$

Conversely, given a total weak harmonic labeling $\ell : V_G \cup E_G \to \mathbb{Z}$ over a simple graph G we can define a weak harmonic labeling over the multigraph G_ℓ where $V_{G_\ell} = V_G$ and $m_{G_\ell}(v, w) = \ell(\{v, w\})$. View in this fashion, weak harmonic labelings of simple graphs are a particular case of total weak harmonic labelings of simple graphs.

Total weak harmonicity is naturally much less restrictive than weak harmonicity. Any finite simple graph G which admits a weak harmonic labeling in particular admits a bijective vertex-labeling $\ell : V_G \to [0, n-1] \cap \mathbb{Z}$ such that

$$\min_{w \in N_v(G)} \{\ell(w)\} < \ell(v) < \max_{u \in N_v(G)} \{\ell(u)\} \tag{3}$$

for every $v \in V_G \setminus S_G$. Algorithm 5 produces a total weak harmonic labeling from any labeling ϕ fulfilling (3) on a finite simple graph G. It makes use of the following

Remark 6.9. If $\mathcal{M} = (A, m)$ is a finite multiset and $x \in \mathcal{M}$ is neither the maximum or minimum of $\mathcal{M}$ then we can "correct" the multiplicities of the elements of $\mathcal{M}$ so $av(\mathcal{M}) = x$. Indeed, if $x > av(\mathcal{M})$ then letting $s = \min_{y \in \mathcal{M}}\{y\}$ and

$$m'(y) = \begin{cases} m(y) \cdot m(s) \cdot (x - s) & y \neq s, x \\ m(s) \cdot |\sum_{z \neq s} (x - z)m(z)| & y = s \end{cases}$$

we readily see that $\mathcal{M}' = (A, m')$ is an harmonic multiset of $\mathbb{Z}$. The case $x < av(\mathcal{M})$ is analogous. Additionally, note that multiplying the multiplicities of every element in an harmonic multiset of $\mathbb{Z}$ by a fixed positive integer does not alter its harmonicity.

Fig. 12 shows examples of total weak harmonic labelings obtained from Algorithm 5 to some complete graphs with two leaves added.

7. Open questions and future work

Next is a short list of possible future directions and open problems of the theory of weak harmonic labelings.

(1) As it was pointed out by a reviewer, the examples of harmonic labelings obtained from finite weak models in this article enjoy a lot of symmetries. Can a larger class of examples be produced based on this regularity? Is there a way to obtain an harmonic labeling from (the weak labeling) of $K_{1,n}$ for every even n (see Fig. 5)?
(2) Can a given infinite graph admit two different harmonic labelings? Can finite weak models help construct examples of such labelings? This question also arose from a reviewer's comment.
(3) We have shown in Section 3 that many of the examples of weak labelings onto $\mathbb{Z}$ built from finite models belonged to the family $\mathcal{P}$ (see Proposition 3.4). Can all weakly labeled graphs onto $\mathbb{Z}$ which are (weakly labeled) subgraphs of P_B for some $B \subset \mathcal{B}$ be characterized?
(4) In Section 5 we provided quantitative evidence of the efficiency of using Theorem 4.6 to compute weakly labeled finite graphs. It would be interesting to produce concrete efficient implementations to perform these calculations.
(5) Harmonic subsets of $\mathbb{Z}$ are a rather interesting class of subsets of $\mathbb{Z}$. Can they all be characterized/built efficiently? If not, what can we say about its cardinality and/or distribution according to average or size?

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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