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GEOMETRY OF THE PROJECTIVE UNITARY GROUP OF A C^*-ALGEBRA

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Abstract. Let $\mathcal{A}$ be a C^*-algebra with a faithful state φ. It is proved that the projective unitary group $\mathbb{P}\mathcal{U}_\mathcal{A}$ of $\mathcal{A}$,

$$\mathbb{P}\mathcal{U}_\mathcal{A} = \mathcal{U}_\mathcal{A}/\mathbb{T}.1,$$

($\mathcal{U}_\mathcal{A}$ denotes the unitary group of $\mathcal{A}$) is a C^∞-submanifold of the Banach space $\mathcal{B}_s(\mathcal{A})$ of bounded operators acting in $\mathcal{A}$, which are symmetric for the φ-inner product, and are usually called symmetrizable linear operators in $\mathcal{A}$.

A quotient Finsler metric is introduced in $\mathbb{P}\mathcal{U}_\mathcal{A}$, following the theory of homogeneous spaces of the unitary group of a C^*-algebra. Curves of minimal length with any given initial conditions are exhibited. Also it is proved that if $\mathcal{A}$ is a von Neumann algebra (or more generally, an algebra where the unitary group is exponential) two elements in $\mathbb{P}\mathcal{U}_\mathcal{A}$ can be joined by a minimal curve.

In the case when $\mathcal{A}$ is a von Neumann algebra with a finite trace, these minimality results hold for the quotient of the metric induced by the p-norm of the trace ($p \geq 2$), which metrizes the strong operator topology of $\mathbb{P}\mathcal{U}_\mathcal{A}$.

1. Introduction

Let $\mathcal{A}$ be a unital C^*-algebra with norm $\| \ \|_\infty$ and with a faithful state φ. We shall study here the *projective unitary space* of $\mathcal{A}$,

$$\mathbb{P}\mathcal{U}_\mathcal{A} = \mathcal{U}_\mathcal{A}/\mathbb{T}.1,$$

where $\mathcal{U}_\mathcal{A}$ is the unitary group of $\mathcal{A}$. $\mathcal{U}_\mathcal{A}$ is a Banach–Lie group whose Banach–Lie algebra is $\mathcal{A}_{ah} = \{x \in \mathcal{A} : x^* = -x\}$. We shall consider $\mathcal{A}$ represented in the Hilbert space $\mathcal{L}^2 = L^2(\mathcal{A}, \varphi)$, via the GNS representation induced by φ. Elements $x \in \mathcal{A}$ will also be regarded as elements of $\mathcal{L}^2$ with norm $\|x\|_2 = \varphi(x^*x)^{1/2}$. As is usual notation, if $\xi, \eta \in \mathcal{L}^2$, $\xi \otimes \eta$ will denote the rank one operator acting in $\mathcal{L}^2$: $\xi \otimes \eta(\nu) = \langle \nu, \eta\rangle \xi$, and in particular if $x, y, a \in \mathcal{A}$, $x \otimes y(a) = \varphi(y^*a)x$.

Let

$$\mathcal{B}_s(\mathcal{A}) = \mathcal{B}_{s,\varphi}(\mathcal{A}) = \{T \in \mathcal{B}(\mathcal{A}) : \varphi(y^*T(x)) = \varphi(T(y)^*x) \text{ for all } x, y \in \mathcal{A}\}.$$

Although this space depends also on φ, we keep the notation $\mathcal{B}_s(\mathcal{A})$ for the sake of simplicity. These operators acting in $\mathcal{A}$, are known as symmetrizable operators in the literature (see the papers by M.G. Krein [9] and P. Lax [10]). They are

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characterized as operators in $\mathcal{A}$ which extend to bounded symmetric operators in $\mathcal{L}^2$. $\mathcal{B}_s(\mathcal{A})$ is a closed subspace of $\mathcal{B}(\mathcal{A})$, the Banach space of all bounded linear operators acting in $\mathcal{A}$.

There is a natural one to one map

$$\mathbb{P}\mathcal{U}_{\mathcal{A}} \to \{u \otimes u : u \in \mathcal{U}_{\mathcal{A}}\} \subset \mathcal{B}_s(\mathcal{A}), \quad [u] \mapsto u \otimes u.$$

In this paper it is shown that this map is a homeomorphism, if $\mathbb{P}\mathcal{U}_{\mathcal{A}}$ is considered with the quotient topology, and the right hand set with the usual norm of $\mathcal{B}(\mathcal{A})$. The set $\{u \otimes u : u \in \mathcal{U}_{\mathcal{A}}\}$ is shown to be a complemented submanifold of $\mathcal{B}_s(\mathcal{A})$. Thus $\mathbb{P}\mathcal{U}_{\mathcal{A}}$ can be regarded as a submanifold of this Banach space. The differentiable structure induced in $\mathbb{P}\mathcal{U}_{\mathcal{A}}$ is the same as the usual quotient differentiable structure [5], and thus is independent of the choice of φ.

A Finsler structure is introduced in $\mathbb{P}\mathcal{U}_{\mathcal{A}}$, following the theory of homogeneous unitary spaces $\mathcal{U}_{\mathcal{A}}/\mathcal{U}_{\mathcal{B}}$ ($\mathcal{B}$ a unital sub-C^*-algebra of $\mathcal{A}$) of Durán, Mata-Lorenzo and Recht [6, 7]. The tangent spaces are endowed with a $\mathcal{U}_{\mathcal{A}}$-invariant quotient norm. Using general results of this theory, applied to this particular case in which the subalgebra $\mathcal{B} = \mathbb{C}.1$, one obtains existence of minimal curves with given initial data, and in the case when $\mathcal{A}$ is a von Neumann algebra, existence of curves joining any given pair of points in $\mathbb{P}\mathcal{U}_{\mathcal{A}}$. The minimal curves are of the form

$$\gamma(t) = [ue^{itx}] \simeq ue^{itx} \otimes ue^{itx},$$

for $u \in \mathcal{U}_{\mathcal{A}}$ and $x^* = x$ with $\|x\| \leq \pi$. They remain minimal for $|t| \leq 1$.

The case when $\mathcal{A}$ is a von Neumann algebra with a finite trace is considered in the last section. It is shown that these curves γ are also minimal for the quotient p-norms in $T\mathbb{P}\mathcal{U}_{\mathcal{A}}$, for $p \geq 2$. These weaker norms metrize the strong operator topology in $\mathbb{P}\mathcal{U}_{\mathcal{A}}$

2. Regular structure

Consider the fibration

$$\mathcal{U}_{\mathcal{A}} \to \mathbb{P}\mathcal{U}_{\mathcal{A}}, \quad u \mapsto [u].$$

More generally, the smooth left action of $\mathcal{U}_{\mathcal{A}}$ on $\mathbb{P}\mathcal{U}_{\mathcal{A}}$, $w \cdot [u] = [wu]$ induces the submersions

$$\pi_{[u]} : \mathcal{U}_{\mathcal{A}} \to \mathbb{P}\mathcal{U}_{\mathcal{A}}, \quad \pi_{[u]}(w) = [wu].$$

Let us denote by $\delta_{[u]} = d(\pi_{[u]})_1$. The isotropy groups of the action are

$$\pi_{[u]}^{-1}([u]) = \{v \in \mathcal{U}_{\mathcal{A}} : [vu] = [u]\} = \mathbb{T} \cdot 1,$$

and therefore the isotropy Banach–Lie algebras are $i\mathbb{R} \cdot 1$ at every $[u] \in \mathbb{P}\mathcal{U}_{\mathcal{A}}$. In particular, since the tangent space $(T\mathcal{U}_{\mathcal{A}})_1$ of $\mathcal{U}_{\mathcal{A}}$ at 1 is $\mathcal{A}_{as}$, the epimorphism $\delta_{[u]}$ induces a natural isomorphism

$$(T\mathbb{P}\mathcal{U}_{\mathcal{A}})_{[u]} = \mathcal{A}_{as}/i\mathbb{R}.$$

Let us prove that $\mathbb{P}\mathcal{U}_{\mathcal{A}}$ has a submanifold structure. To this effect, we shall use the bijection

$$\mathbb{P}\mathcal{U}_{\mathcal{A}} = \mathcal{U}_{\mathcal{A}}/\mathbb{T} \longleftrightarrow \{u \otimes u : u \in \mathcal{U}_{\mathcal{A}}\} \subset \mathcal{B}_s(\mathcal{A}), \quad [u] \longleftrightarrow u \otimes u.$$

By means of this map, we can regard $\mathbb{P}\mathcal{U}_{\mathcal{A}}$ as a subset of a Banach space.

Lemma 2.1. *The map*

$$\mathbb{P}\mathcal{U}_{\mathcal{A}} \to \{u \otimes u : u \in \mathcal{U}_{\mathcal{A}}\}, \quad [u] \mapsto u \otimes u$$

is a homeomorphism, when the right hand set is considered with the norm topology of $\mathcal{B}_s(\mathcal{A})$.

Proof. The map $\mathcal{U}_{\mathcal{A}} \to \{u \otimes u : u \in \mathcal{U}_{\mathcal{A}}\}$, $u \mapsto u \otimes u$ is continuous, and induces the above bijection from the quotient $\mathbb{P}\mathcal{U}_{\mathcal{A}} = \mathcal{U}_{\mathcal{A}}/\mathbb{T}$, which is therefore continuous.

Let us see that the inverse is continuous. The map is equivariant for the transitive actions of $\mathcal{U}_{\mathcal{A}}$ on both spaces:

$$w \cdot [u] = [wu] \mapsto wu \otimes wu = L_w(u \otimes u)L_{w^*}.$$

Thus it suffices to prove that the inverse map is continuous at $1 \otimes 1$. Suppose that the $u_n \in \mathcal{U}_{\mathcal{A}}$ satisfy

$$u_n \otimes u_n \to 1 \otimes 1 \quad \text{as } n \to \infty.$$

Then evaluating at 1, $\overline{\varphi(u_n)}u_n \to 1$. Thus

$$\varphi(\overline{\varphi(u_n)}u_n) = |\varphi(u_n)|^2 \to 1.$$

Then

$$\frac{\overline{\varphi(u_n)}}{|\varphi(u_n)|}u_n \to 1,$$

i.e., there exist $\lambda_n = \frac{\overline{\varphi(u_n)}}{|\varphi(u_n)|}$ with $|\lambda_n| = 1$ such that $\lambda_n u_n \to 1$, i.e., $[u_n] \to [1]$. $\square$

In particular, this implies that the topological structure of the set $\{u \otimes u : u \in \mathcal{U}_{\mathcal{A}}\}$ does not depend on the state φ:

Corollary 2.2. *Let φ, ψ be faithful states in $\mathcal{A}$, and denote by $\mathcal{L}^2_\varphi, \mathcal{L}^2_\psi$ their GNS Hilbert spaces. Then the sets*

$$\{u \otimes_\varphi u : u \in \mathcal{U}_{\mathcal{A}}\} \subset \mathcal{B}(\mathcal{L}^2_\varphi) \quad \textit{and} \quad \{u \otimes_\psi u : u \in \mathcal{U}_{\mathcal{A}}\} \subset \mathcal{B}(\mathcal{L}^2_\psi)$$

are homeomorphic (with the corresponding norm topologies). Specifically, the map

$$u \otimes_\varphi u \mapsto u \otimes_\psi u$$

is a homeomorphism.

Remark 2.3. Note that the set $\{u \otimes u : u \in \mathcal{U}_{\mathcal{A}}\}$ is a set of rank one projections in $\mathcal{B}(\mathcal{A})$ (or in $\mathcal{B}(\mathcal{L}^2)$ as well): indeed,

$$\langle u, u \rangle = \varphi(u^* u) = 1.$$

To prove that $\mathbb{P}\mathcal{U}_{\mathcal{A}}$ is a submanifold of $\mathcal{B}_s(\mathcal{A})$, we shall use the following lemma, which was proved in [11].

Lemma 2.4. *Let G be a Banach–Lie group acting smoothly on a Banach space X. For a fixed $x_0 \in X$, denote by $\pi_{x_0} : G \to X$ the smooth map $\pi_{x_0}(g) = g \cdot x_0$. Suppose that*

(1) π_{x_0} *is an open mapping, regarded as a map from* G *onto the orbit* $\{g\cdot x_0 : g\in G\}$ *of* x_0 *(with the relative topology of* X*).*
(2) *The differential* $d(\pi_{x_0})_1 : (TG)_1 \to X$ *splits: its nullspace and range are closed complemented subspaces.*

Then the orbit $\{g\cdot x_0 : g\in G\}$ *is a smooth submanifold of* X*, and the map*

$$\pi_{x_0} : G \to \{g\cdot x_0 : g\in G\}$$

is a smooth submersion.

Theorem 2.5. $\mathbb{P}\mathcal{U}_{\mathcal{A}}$ *is a closed complemented* C^∞*-submanifold of* $\mathcal{B}_s(\mathcal{A})$ *and the map*

$$\pi : \mathcal{U} \mapsto \mathbb{P}\mathcal{U}_{\mathcal{A}}, \quad \pi(u) = u\otimes u$$

is a C^∞*-submersion.*

Proof. Let us prove first that $\{u\otimes u : u\in \mathcal{U}_{\mathcal{A}}\}$ is a closed subset of $\mathcal{B}_s(\mathcal{A})$. Suppose that $u_n\otimes u_n \to T$ in $\mathcal{B}_s(\mathcal{A})$. Evaluating at 1 one obtains that $\varphi(u_n^*)u_n = \overline{\varphi(u_n)}u_n$ is convergent in $\mathcal{A}$. Since $|\varphi(u_n)| \le \varphi(u_n^*u_n)^{1/2} = 1$, there is a convergent subsequence $\varphi(u_{n_k})$. Then u_{n_k} converges to a unitary $u\in\mathcal{A}$. Therefore $u_n\otimes u_n$ converges to $u\otimes u$.

Fix $1\otimes 1\in \mathbb{P}\mathcal{U}_{\mathcal{A}}$. We shall construct a continuous local cross section for

$$\pi = \pi_{1\otimes 1} : \mathcal{U}_{\mathcal{A}} \to \mathbb{P}\mathcal{U}_{\mathcal{A}}, \quad \pi(u) = u\otimes u = L_u(1\otimes 1)L_{u^*},$$

near $1\otimes 1$. Cross sections near other points are obtained by translation with the group action. Consider the open set

$$\mathcal{V} = \{u\otimes u : (u\otimes u)(1\otimes 1)\neq 0\}.$$

It is clear that $\mathcal{V}$ is an open neighbourhood of $1\otimes 1$ in $\mathbb{P}\mathcal{U}_{\mathcal{A}}$. Note that $(u\otimes u)(1\otimes 1)\neq 0$ means that $\varphi(u^*)u\otimes 1\neq 0$, i.e., $\varphi(u)\neq 0$. Put

$$\mu : \mathcal{V} \to \mathcal{U}_{\mathcal{A}}, \quad \mu(u\otimes u) = \frac{\varphi(u^*)}{|\varphi(u)|}u.$$

The map μ is well defined: if $u\otimes u = w\otimes w$ then $w = \lambda u$ with $\lambda\in\mathbb{T}$. Thus

$$\frac{\varphi(w^*)}{|\varphi(w)|}w = \frac{\overline{\lambda}\varphi(u^*)}{|\varphi(u)|}\lambda u = \frac{\varphi(u^*)}{|\varphi(u)|}u.$$

Also $\mu(1\otimes 1) = 1$. It is clear that μ is a cross section for π. Let us prove that μ is the restriction of a map $\tilde\mu$ defined on a neighbourhood of $1\otimes 1$ in $\mathcal{B}_s(\mathcal{A})$. Namely

$$\tilde\mu : \tilde{\mathcal{V}} = \{T\in\mathcal{B}_s(\mathcal{A}) : T(1)\neq 0\} \to \mathcal{A}, \quad \tilde\mu(T) = |\varphi(T(1))|^{-1/2}T(1).$$

Indeed, if $T = u\otimes u$, then $T(1) = \varphi(u^*)u$ and $\varphi(T(1)) = |\varphi(u)|^2$. It is clear that $\tilde\mu$ is continuous. Therefore μ is continuous. Thus π is open. A straightforward computation shows that the differential of π at 1 is (to differentiate π we regard it as a map valued in $\mathcal{B}_s(\mathcal{A})$)

$$\delta = d\pi_1 : \mathcal{A}_{as} \to \mathcal{B}_s(\mathcal{A}), \quad \delta(a) = a\otimes 1 + 1\otimes a.$$

The nullspace of this map is $i\mathbb{R}\cdot 1$. Indeed, it is clear that $i\mathbb{R}\cdot 1 \subset N(\delta)$. If $\delta(a)=0$, then in particular $0=(a\otimes 1+1\otimes a)(1)=1+\varphi(a^*)1$, i.e., $a=-\overline{\varphi(a)}1$. Thus $N(\delta)$ is complemented.

To prove that $R(\delta)=\{a\otimes 1+1\otimes a : a\in\mathcal{A}_{as}\}$ is complemented in $\mathcal{B}_s(\mathcal{A})$, note that the map $\tilde{\mu}$ is C^∞ in $\tilde{\mathcal{V}}$. Denote by $\rho = d\tilde{\mu}_{1\otimes 1}$,

$$\rho : \mathcal{B}_s(\mathcal{A}) \to \mathcal{A}.$$

For u close to 1 (in order that $\varphi(u)\neq 0$),

$$\pi\circ\tilde{\mu}\circ\pi(u)=\pi\circ\tilde{\mu}(u\otimes u)=\pi(u),$$

i.e., $\pi\circ\tilde{\mu}\circ\pi=\pi$ near 1. Differentiating this identity at 1, we get

$$\delta\rho\delta=\delta.$$

In particular, $\delta\rho$ is an idempotent operator acting in the (real) Banach space $\mathcal{B}_s(\mathcal{A})$, whose range is the range of δ. Then $R(\delta)$ is complemented, and the proof is complete. □

It is clear that $\mathbb{P}\mathcal{U}_\mathcal{A}$ is a group. Let us check that the group operations are smooth.

Proposition 2.6. *$\mathbb{P}\mathcal{U}_\mathcal{A}$ is a C^∞ Banach–Lie group.*

Proof. Consider first the product map

$$\Pi : \mathcal{U}_\mathcal{A}\times\mathcal{U}_\mathcal{A}\to\mathcal{U}_\mathcal{A}, \quad \Pi(u,w)=uw.$$

This map induces the product map on the quotient

$$\tilde{\Pi} : \mathbb{P}\mathcal{U}_\mathcal{A}\times\mathbb{P}\mathcal{U}_\mathcal{A}\to\mathbb{P}\mathcal{U}_\mathcal{A}, \quad \tilde{\Pi}([u],[w])=[u][w].$$

The fact that the product is defined in the quotient, i.e. that $[uw]=[u][w]$, means that

$$\pi\circ\Pi=\tilde{\Pi}\circ(\pi\times\pi).$$

By the above theorem π is a submersion, and therefore has local C^∞ cross sections. Let $\mu_{[u_0]}$ and $\mu_{[w_0]}$ be cross sections for π defined on neighbourhoods $\mathcal{V}_{[u_0]}$, $\mathcal{V}_{[w_0]}$ of $[u_0]$, $[w_0]$, respectively. Then $\mu_{[u_0]}\times\mu_{[w_0]}$ is a cross section for $\pi\times\pi$ defined on $\mathcal{V}_{[u_0]}\times\mathcal{V}_{[w_0]}$, which is a neighbourhood for $([u_0],[w_0])$ in $\mathbb{P}\mathcal{U}_\mathcal{A}\times\mathbb{P}\mathcal{U}_\mathcal{A}$. Then, in this neighbourhood, one has

$$\tilde{\Pi}=\pi\circ\Pi\circ(\mu_{[u_0]}\times\mu_{[w_0]}),$$

which is C^∞. The proof for the inversion map is similar. □

With a similar argument, we can prove that the differentiable structure of $\mathbb{P}\mathcal{U}_\mathcal{A}$, defined in terms of φ, does not depend on the choice of the state φ. We use the notation of Corollary 2.2.

Proposition 2.7. *Let φ,ψ be faithful states in $\mathcal{A}$. Then the map*

$$\{u\otimes_\varphi u : u\in\mathcal{U}_\mathcal{A}\}\to\{u\otimes_\psi u : u\in\mathcal{U}_\mathcal{A}\}, \quad u\otimes_\varphi u\mapsto u\otimes_\psi u$$

is a diffeomorphism.

Proof. Let $\pi_\varphi : \mathcal{U}_\mathcal{A} \to \{u \otimes_\varphi u : u \in \mathcal{U}_\mathcal{A}\}$ and $\pi_\psi : \mathcal{U}_\mathcal{A} \to \{u \otimes_\psi u : u \in \mathcal{U}_\mathcal{A}\}$, and denote by $\theta : \{u \otimes_\varphi u : u \in \mathcal{U}_\mathcal{A}\} \to \{u \otimes_\psi u : u \in \mathcal{U}_\mathcal{A}\}$. Then it is clear that

$$\theta \pi_\varphi = \pi_\psi.$$

Since π_φ is a submersion, it has local C^∞-cross sections μ_φ near every point. Thus locally,

$$\theta = \pi_\psi \mu_\varphi,$$

and therefore θ is C^∞. □

Example 2.8. Suppose that $\mathcal{B}$ is a C^*-algebra with no unit, and let $\tilde{\mathcal{B}} = \mathcal{A}$ be its smallest unitization (i.e., $\mathcal{B}$ is a maximal bilateral ideal and a hyperplane of $\mathcal{A}$). Then it is clear that the projective unitary group $\mathbb{P}\mathcal{U}_\mathcal{A}$ is isomorphic (as a Banach–Lie group) to the group

$$G_\mathcal{B} = \{u \in \mathcal{U}_\mathcal{A} : u - 1 \in \mathcal{B}\}.$$

The C^∞ group isomorphism is induced by the inclusion $G_\mathcal{B} \subset \mathcal{U}_\mathcal{A}$. Indeed, since $\mathcal{B}$ has no unit, elements in $\mathcal{U}_\mathcal{A}$ are of the form $\lambda 1 + b$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$ and $b \in \mathcal{B}$. The map $\mathcal{A} \to \mathbb{C}$, $\lambda 1 + b \mapsto \lambda$ is a multiplicative functional, thus C^∞. Then the map

$$\mathcal{U}_\mathcal{A} \to G_\mathcal{B}, \quad \lambda 1 + b \mapsto 1 + \frac{1}{\lambda} b$$

is C^∞ and a group homomorphism, which induces the inverse of the map induced by the inclusion. In the case $\mathcal{B} = \mathcal{K}(\mathcal{H})$ the algebra of compact operators, the group $G_{\mathcal{K}(\mathcal{H})}$ is one of the *classical* Banach–Lie groups, sometimes called the unitary Fredholm group.

3. Metric properties

The following facts are well known.

Remark 3.1. If one endows the unitary group $\mathcal{U}_\mathcal{A}$ with the Finsler metric which consists of the usual norm of $\mathcal{A}$ at every tangent space, the metric geodesics (short curves) of $\mathcal{U}_\mathcal{A}$ which start at a given u are of the form

$$\mu(t) = ue^{itx},$$

for any $x^* = x$ (which we suppose normalized $\|x\|_\infty = 1$) and remain of minimal length for $|t| \le \pi$.

If $\mathcal{A}$ is a von Neumann algebra, any pair of unitaries u_1, u_2 in $\mathcal{U}_\mathcal{A}$ can be joined by such a (minimal) curve, which is unique if $\|u_1 - u_2\|_\infty < 2$.

The following result is a simple case in the problem of finding minimal elements in C^*-algebra inclusions (see for instance [3] and references therein).

Remark 3.2. Given $x = x^* \in \mathcal{A}$, there exists $r = r(x) \ge 0$, such that

$$\|x - r\| = \min\{\|x + t\| : t \in \mathbb{R}\}.$$

The existence of r follows from a compactness argument. Also note that

$$r(x) = \frac{1}{2}\left\{ \max_{\xi \in \mathcal{L}^2,\, \|\xi\|=1} \langle x\xi, \xi\rangle + \min_{\xi \in \mathcal{L}^2,\, \|\xi\|=1} \langle x\xi, \xi\rangle \right\},$$

which is the midpoint in the spectrum $\sigma(x)$ of x.

Definition 3.3. We shall call the element $x - r(x)$ the *minimal lifting* of x.

The tangent space $(T\mathbb{P}\mathcal{U}_\mathcal{A})_{[u]}$ at $[u] \in \mathbb{P}\mathcal{U}_\mathcal{A}$ is given by

$$(T\mathbb{P}\mathcal{U}_\mathcal{A})_{[u]} = \{u \otimes z + z \otimes u : z \in u\mathcal{A}_{as} = \mathcal{A}_{as}u\}.$$

Indeed, let $u(t)$ be a smooth curve in $\mathcal{U}_\mathcal{A}$ with $u(0) = u$ and $\dot{u}(0) = z$ (note that differentiating $u^*(t)u(t) = 1$ at $t = 0$, one gets $z^*u + u^*z = 0$, i.e., $u^*z, zu^* \in \mathcal{A}_{as}$). Then differentiating $u(t) \otimes u(t)$ at $t = 0$ one obtains that tangent vectors at $[u]$ (identified with $u \otimes u$) are of the form $z \otimes u + u \otimes z$.

We endow $\mathbb{P}\mathcal{U}_\mathcal{A}$ with the quotient metric of the usual norm of $\mathcal{A}$.

Definition 3.4. If $z \otimes u + u \otimes z \in (T\mathbb{P}\mathcal{U}_\mathcal{A})_{[u]}$, put

$$|z \otimes u + u \otimes z|_{[u]} = \inf\{\|u^*z - it\| : t \in \mathbb{R}\}.$$

The nullspace of

$$d\pi_u : (T\mathcal{U}_\mathcal{A})_u \to (T\mathbb{P}\mathcal{U}_\mathcal{A})_{[u]}$$

is $N(d\pi_u) = i\mathbb{R}u$, i.e., the norm defined here coincides with the quotient norm of $\mathcal{A}_{as}/i\mathbb{R}$.

Remark 3.5. This metric coincides with the metric defined by Durán et al. in [6] and [7] for homogeneous spaces $\mathcal{U}_\mathcal{A}/\mathcal{U}_\mathcal{B}$ of an inclusion $\mathcal{B} \subset \mathcal{A}$ of C^*-algebras (we treat here the particular case $\mathcal{B} = \mathbb{C}1$). In these papers the metric is induced by the action of $\mathcal{U}_\mathcal{A}$ on the quotient: if $[u] \in \mathcal{U}_\mathcal{A}/\mathcal{U}_\mathcal{B}$, put

$$L_{[u]} : \mathcal{U}_a \to \mathcal{U}_\mathcal{A}/\mathcal{U}_\mathcal{B}, \quad L_{[u]}(w) = [uw].$$

The metric defined on $T(\mathcal{U}_\mathcal{A}/\mathcal{U}_\mathcal{B})_{[u]}$ is the quotient norm (defined by $d(L_{[u]})_1$) in $\mathcal{A}_{as}/\mathcal{B}_{as}$. It is easy to see that in the case $\mathcal{B} = \mathbb{C}1$, this is precisely the metric defined above. Therefore one obtains in our case the general results and properties proved in [6] and [7]. For instance, that the metric is invariant by the left action of $\mathcal{U}_\mathcal{A}$ on $\mathbb{P}\mathcal{U}_\mathcal{A}$. Also the main results on existence of minimal geodesics with given initial data [6] or given endpoints [7] apply here. However, the fact that $\mathcal{B} = \mathbb{C}.1$ allows one to prove these facts in a direct way.

Since the map

$$\pi : \mathcal{U}_\mathcal{A} \to \mathbb{P}\mathcal{U}_\mathcal{A}$$

is a submersion, smooth curves in $\mathbb{P}\mathcal{U}_\mathcal{A}$ lift to continuous piecewise smooth curves in $\mathcal{U}_\mathcal{A}$, joining the fibres of the endpoints of the curve in $\mathbb{P}\mathcal{U}_\mathcal{A}$.

The following result was proved in [2]. Let us denote by d_g the metric obtained in $\mathbb{P}\mathcal{U}_\mathcal{A}$ from the Finsler metric defined in Definition 3.4.

Lemma 3.6. *If* $[u], [v] \in \mathbb{P}\mathcal{U}_\mathcal{A}$,

$$d_g([u], [v]) = \inf\{\ell(\Gamma) : \Gamma(t) \in \mathcal{U}_\mathcal{A} \textit{ smooth}, \ [\Gamma] \textit{ joins } [u] \textit{ and } [v]\},$$

where ℓ denotes the length of the curve measured with the usual norm of $\mathcal{A}$.

Theorem 3.7. *Let $[u] \in \mathbb{P}\mathcal{U}_{\mathcal{A}}$ and $z \otimes u + u \otimes z \in (T\mathbb{P}\mathcal{U}_{\mathcal{A}})_{[u]}$, $w^*z \in \mathcal{A}_{as}$, with $|z \otimes w + w \otimes z|_{[u]} \leq \pi$. Then the curve $[\delta]$*

$$[\delta](t) = ue^{itx_0} \otimes ue^{itx_0}$$

for $x_0 = -iz - r(-iz)$ (i.e., the minimal lifting of $z \otimes w + w \otimes z$), has minimal length for the metric from Definition 3.4, for $|t| \leq \pi$.

Proof. In [6], the general case of a quotient $\mathcal{U}_{\mathcal{A}}/\mathcal{U}_{\mathcal{B}}$ was considered, for an inclusion $\mathcal{B} \subset \mathcal{A}$ of arbitrary C^*-algebras. In our particular case $\mathcal{B} = \mathbb{C}$, one has existence and uniqueness of minimal liftings (in general, minimal liftings may not exist or may not be unique).

Since the action of $\mathcal{U}_{\mathcal{A}}$ is isometric, it suffices to consider the case $[u] = [1]$. The curve $[\delta]$ has an obvious lifting $\delta(t) = e^{itx_0}$. Let ω be another curve of unitaries joining the fibers of 1 and v. Since exponentials are short in the unitary group, and the action of $\mathcal{U}_{\mathcal{A}}$ is isometric, we can suppose ω to be of the form $\omega(t) = e^{ity}$. Furthermore, since $[e^{ix_0}] = [e^{iy}]$, we have that $y = x_0 + s_0$. Since x_0 is a minimal lifting,

$$\ell(\delta) = \|x_0\| \leq \|x_0 + s_0\| = \ell(\omega),$$

because $\|x_0\| \leq \pi$. On the other hand, $\|x_0\| = L([\delta])$, and the proof follows. □

Let us return to Example 2.8, of a non unital C^*-algebra $\mathcal{B}$ and $\mathcal{A} = \tilde{\mathcal{B}}$ its minimal unitization.

Example 3.8. The isomorphism

$$\mathbb{P}\mathcal{U}_{\tilde{\mathcal{B}}} \to G_{\mathcal{B}} = \{u \in \mathcal{U}_{\tilde{\mathcal{B}}} : u - 1 \in \mathcal{B}\}$$

induces a metric in $G_{\mathcal{B}}$. Namely, the Banach–Lie algebra of $G_{\mathcal{B}}$ is $\mathcal{B}_{ah}$. If $b \in \mathcal{B}_{ah}$, then the metric induced by the norm of $\mathcal{A}$ is

$$|b|_0 = \inf\{\|b - \lambda 1\| : \lambda \in \mathbb{C}\} = \inf\{\|b - ir1\| : r \in \mathbb{R}\},$$

which is, as we have seen, the midpoint of the spectrum of b. Let us characterize the minimal curves of $G_{\mathcal{B}}$. If $z = \lambda 1 + b \in \tilde{\mathcal{B}}_{ah}$, then

$$e^z = e^\lambda e^b = e^\lambda(1 + b + \frac{1}{2}b^2 + \dots) = e^\lambda 1 + b',$$

where $b' = b + \frac{1}{2}b^2 + \dots \in \mathcal{B}$. Thus the isomorphism $\mathbb{P}\mathcal{U}_{\mathcal{A}}$ sends $[e^z]$ to $\frac{1}{e^\lambda}e^z = e^b$. It follows that for this (midpoint-spectrum) metric, curves

$$\delta(t) = ge^{tb}$$

for $g \in G_{\mathcal{B}}$ and $b \in \mathcal{B}_{ah}$ are minimal for $|t| \leq \frac{\pi}{|b|_0}$. This norm $|\ |_0$ defined in $\mathcal{B}_{ah}$ is equivalent to the usual norm $\|\ \|$. Indeed, it is clear that $|b|_0 \leq \|b\|$. Putting $b = ib'$ with b' selfadjoint,

$$2|b|_0 = \sup \sigma(b') - \inf \sigma(b').$$

Since b' is non invertible ($\mathcal{B}$ is non unital), it must be $\sup \sigma(b') \geq 0$ (otherwise the spectrum of b' would be strictly negative and b' invertible). Then $\inf \sigma(b') \leq 0$, and thus

$$\sup \sigma(b') - \inf \sigma(b') \geq \max\{\sup \sigma(b'), -\inf \sigma(b')\} = \sup_{\lambda \in \sigma(b')} |\lambda| = \|b'\|.$$

Then

$$\frac{1}{2}\|b\| \leq |b|_0 \leq \|b\|.$$

If $\mathcal{A}$ is a von Neumann algebra, one can prove that given two points $[u], [v] \in \mathbb{P}\mathcal{U}_{\mathcal{A}}$, there exists a minimal curve joining them. The existence of minimal curves joining given endpoints which are close was proved in [7], for arbitrary $\mathcal{B} \subset \mathcal{A}$.

Theorem 3.9. *Let $\mathcal{A}$ be a von Neumann algebra. Let $[u], [v] \in \mathbb{P}\mathcal{U}_{\mathcal{A}}$. Then there exists a minimal geodesic $[\delta]$ for the metric from Definition 3.4 ($\delta(t) = ue^{itx_0}$, with x_0 a minimal lifting for $\| \ \|$) such that $[\delta(0)] = [u]$ and $[\delta(1)] = [v]$.*

Proof. There exists $x = x^* \in \mathcal{A}$ such that $v = ue^{ix}$ with $\|x\| \leq \pi$. Let $x_0 = x - r(x)$. Then, since x_0 is a minimal lifting,

$$\|x_0\| \leq \|x\| \leq \pi.$$

Thus $[\delta(y)] = [e^{itx_0}]$ has minimal length between its endpoints for $t \in [0,1]$ by the preceeding theorem. Its endpoints are

$$[\delta(0)] = [u] \quad \text{and} \quad [\delta(1)] = [ue^{itx - r(x)}] = [ve^{itr(x)}] = [v]. \qquad \square$$

Remark 3.10. The result holds with the same proof for C^*-algebras $\mathcal{A}$ such that the unitary group $\mathcal{U}_{\mathcal{A}}$ is exponential (i.e., $\mathcal{U}_{\mathcal{A}} = \exp(\mathcal{A}_{ah})$). For instance, as in Examples 2.8 and 3.8, put $\mathcal{B} = \mathcal{K}(\mathcal{H})$. Then it is well known that

$$G_{\mathcal{K}(\mathcal{H})} = \exp(\mathcal{K}_{ah}(\mathcal{H})).$$

4. Finite von Neumann algebras

For the case when $\mathcal{A}$ is a finite von Neumann algebra with a finite (normal, faithful) trace τ, one can endow the tangent spaces of $\mathcal{U}_{\mathcal{A}}$ with the p-norm $\|x\|_p = \tau(x^*x)^{p/2}$, and one obtains a metric which is equivalent to the p-norm restricted to $\mathcal{U}_{\mathcal{A}}$, which is complete (and metrizes both the weak and strong operator topologies of $\mathcal{U}_{\mathcal{A}}$). For this metric, the same curves μ of Remark 3.1 are minimal, and remain so for $|t| \leq \pi$ if $\|x\|_\infty \leq \pi$. Note that the normalization of the exponent x is done in the usual norm of $\mathcal{A}$. A geodesic joining u_1 and u_2 is unique if $\|u_1 - u_2\|_\infty < 2$ (again, usual norm of $\mathcal{A}$). These facts were proved in [1] for $p \geq 2$, though the author believes it holds for $p \geq 1$ (see [4], where the analogous result was proved for $p \geq 1$, for the usual (infinite) trace of $\mathcal{B}(\mathcal{H})$).

Let $p \geq 2$ and $x^* = x \in \mathcal{A}$. Then there exists a unique $r = r(x,p) \in \mathbb{R}$ such that

$$\|x - r\|_p = \min\{\|x + t\|_p : t \in \mathbb{R}\}.$$

If $p = 2$, $r = \tau(x)$. In general, the map

$$f(t) = \|x + t\|_p^p, \quad t \in \mathbb{R},$$

is strictly convex (this follows, for instance, from the uniform convexity of the p-norm [8]), and tends to $+\infty$ if $|t| \to \infty$. Thus it has a (unique) global minimum.

The minimality results of the previous section hold for the p-norms. Let us define

Definition 4.1. For $x = x^* \in \mathcal{M}$, we call the element $x - r(x,p)$ the *p-minimal lifting* of x.

Definition 4.2. If $z \otimes u + u \otimes z \in (T\mathbb{P}\mathcal{U}_{\mathcal{A}})_{[u]}$, put

$$|z \otimes u + u \otimes z|_{[u],p} = \inf\{\|u^*z - it\|_p : t \in \mathbb{R}\},$$

the *p-quotient metric* on $\mathcal{PU}_{\mathcal{A}}$.

Lemma 3.6 was proved in [2] for the p-norms, for $2 \leq p < \infty$. Therefore the analogue of Theorem 3.7 can be proved in a similar fashion:

Theorem 4.3. *Let $\mathcal{A}$ be a finite von Neumann algebra, $[u] \in \mathbb{P}\mathcal{U}_{\mathcal{A}}$ and $z \otimes u + u \otimes z \in (T\mathbb{P}\mathcal{U}_{\mathcal{A}})_{[u]}$, $w^*z \in \mathcal{A}_{as}$, with $|z \otimes w + w \otimes z|_{[u]} \leq \pi$. Then the curve $[\delta]$,*

$$[\delta](t) = ue^{itx_0} \otimes ue^{itx_0}$$

for $x_0 = -iz - r(-iz,p)$ (i.e., the minimal lifting of $z \otimes w + w \otimes z$), has minimal length for the p-quotient metric in Definition 4.2, for $|t| \leq \pi$.

And therefore one has also the analogue of Theorem 3.9:

Theorem 4.4. *Let $\mathcal{A}$ be a finite von Neumann algebra. Let $[u],[v] \in \mathbb{P}\mathcal{U}_{\mathcal{A}}$. Then there exists a minimal geodesic $[\delta]$ for the metric from Definition 4.2 for any even p, ($\delta(t) = ue^{itx_0}$, with x_0 a minimal lifting for $\|\ \|_p$) such that $[\delta(0)] = [u]$ and $[\delta(1)] = [v]$.*

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