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Geometric significance of Toeplitz kernels

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Let L^2 be the Lebesgue space of square-integrable functions on the unit circle. We show that the injectivity problem for Toeplitz operators is linked to the existence of geodesics in the Grassmann manifold of L^2. We also investigate this connection in the context of restricted Grassmann manifolds associated to p-Schatten ideals and essentially commuting projections.

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1. Introduction

Let L^p be the usual Lebesgue spaces of complex-valued functions on the unit circle $\mathbb{T}$. The Grassmann manifold of L^2 is the set of all closed subspaces of L^2. This paper studies the relation between geodesics on the Grassmann manifold of L^2 and the injectivity problem for Toeplitz operators.

To explain this relation, let H^2 be the Hardy space of the unit circle. Recall that the injectivity problem for Toeplitz operators consists in looking for those symbols $\varphi \in L^\infty$ such that the Toeplitz operator T_φ is injective. We relate it to the problem of finding a geodesic on the Grassmann manifold of L^2 which joins two subspaces of the form φH^2 and ψH^2, where φ, ψ are invertible functions in L^∞. More precisely, we will prove that such a geodesic exists if and only if the Toeplitz operator $T_{\varphi\psi^{-1}}$ and its adjoint both have trivial kernel. Furthermore, we will see that these statements are also equivalent to the existence of a minimizing geodesic joining the given subspaces.

The Grassmann manifold of an abstract Hilbert space (i.e. the set consisting of all the closed subspaces) may be identified with the bounded self-adjoint projections. It is an infinite dimensional homogeneous space which can be endowed with a Finsler metric by using the operator norm on each tangent space. Although it is complete with the corresponding rectifiable distance, there are subspaces in the same connected component that cannot be joined by a geodesic (see e.g. [1]). This means that the Hopf–Rinow theorem fails for this manifold. Nevertheless, much information of its geodesics and their minimizing properties are known. The first results date back to the works [21,13,29]; all in the more general framework of self-adjoint projections in C^*-algebras. More recently, there has been progress about the structure of the geodesics in several Grassmann manifolds defined by imposing additional conditions on the subspaces; see for instance [5,3,6] for restricted Grassmann manifolds and [4] for the Lagrangian Grassmann manifold.

In this paper, we turn to a more concrete setting by taking the Hilbert space L^2. This allows us to study the interplay between geodesics, functional spaces and operator theory. In contrast to the invertibility problem for Toeplitz operators, little attention has been paid in the literature to the injectivity problem until recent years. Except for the works of [12,22], the problem remained untreated until the recent works [23–25] (see also the survey [19]). Apart from being an interesting problem in operator theory, in these latter articles there are relevant applications to harmonic analysis, complex analysis and mathematical physics.

The structure of this paper is as follows. In Section 2 we give classical results on Hardy spaces, Toeplitz and Hankel operators to make the article reasonably self-contained. In Section 3 we prove the aforementioned relation between geodesics of the Grassmann manifold of L^2 and the injectivity problem (Theorem 3.4). Then, this result is used to derive an inequality involving the reduced minimum modulus of Toeplitz operators and the norm of a commutator (Theorem 3.8).

In Section 4 we deal with the compact restricted Grassmannian (or Sato Grassmannian). This is a well-known Banach manifold related to KdV equations and loop groups (see [32,33]). We need to consider the following two uniform sub-algebras of L^∞: the continuous functions C and the usual Hardy space H^∞. We show that a subspace φH^2 belongs to the compact restricted Grassmannian if and only if φ is an invertible function in the Sarason algebra $H^\infty + C$. This is the least nontrivial closed sub-algebra lying between H^∞ and L^∞; it has also been extensively studied [8,16,31]. The existence of geodesics in the restricted Grassmannian between two subspaces φH^2 and ψH^2, φ, ψ invertible functions in $H^\infty + C$, depends only on the index of these functions (Theorem 4.2). We also examine when a subspace φH^2 can be written as $\varphi H^2 = gH^2$, where g is a continuous unimodular function. These results can be carried out also in the setting of restricted Grassmannians associated to p-Schatten ideals by using the notion of Krein algebras defined in [9].

Section 5 focuses on shift-invariant subspaces of H^2. Each shift-invariant subspace can be expressed as φH^2, where φ is an inner function. We prove that the canonical factorization of φ determines the class where the subspace φH^2 belongs (Theorem 5.1). Based on the results on the injectivity problem mentioned above, we provide examples showing the existence or non-existence of geodesics between shift-invariant subspaces.

2. Background

For $1 \leq p \leq \infty$, $L^p = L^p(\mathbb{T})$ denotes the usual Lebesgue space of complex valued functions defined on the unit circle $\mathbb{T}$. The Hardy space H^p ($1 \leq p < \infty$) is the space of all analytic functions f on the disk $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ for which

$$\|f\|_{H^p} := \left(\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p} < \infty.$$

The space of all bounded analytic functions on $\mathbb{D}$ with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$ is the Hardy space H^∞. Functions in Hardy spaces have non-tangential limits a.e., a fact which is used to isometrically identify these spaces with

$$H^p = \{ f \in L^p : \int_0^{2\pi} f(e^{it})\overline{\chi_n(e^{it})} \, dt = 0, \, n < 0 \}.$$

Here $(\chi_k)_{k\in\mathbb{Z}}$ denotes the orthonormal basis of L^2 given by $\chi_k(e^{it}) = e^{ikt}$. We shall mostly use this representation of Hardy spaces as functions defined on $\mathbb{T}$ and deal with the values $p = 2, \infty$. In particular, H^2 is a closed subspace of the Hilbert space L^2 and H^∞ is a closed sub-algebra of L^∞. For background and notational purposes, our main references for this paper are the books by Douglas, Nikol'skiĭ and Pavlović [26,27,16,28].

A function $f \in H^2$ is called *inner* if $|f(e^{it})| = 1$ a.e. on $\mathbb{T}$. A function $f \in H^2$ is *outer* if $\overline{\text{span}}\{ f\chi_n : n \geq 0 \} = H^2$. For each $f \in H^2$, $f \neq 0$, there exist an inner function f_{inn}

and an outer function $f_{out} \in H^2$ such that $f = f_{inn} f_{out}$. This is called the *inner-outer factorization*, and it is unique up to a multiplicative constant.

The inner function can be further factorized. For each $a \in \mathbb{D} \setminus \{ 0 \}$, a Blaschke factor is given by

$$b_a(z) = \frac{\overline{a}}{|a|} \frac{a - z}{1 - \overline{a} z}, \quad z \in \mathbb{D}.$$

When $a = 0$, set $b_0(z) = z$. A Blaschke product is a function of the form

$$b(z) = \prod_{j=1}^{n} b_{a_j}(z), \quad z \in \mathbb{D},$$

where $1 \leq n \leq \infty$. In the case where $n = \infty$, the infinite Blaschke product is convergent on compact subsets of $\mathbb{D}$ if the sequence $\{ a_j \} \subseteq \mathbb{D}$ satisfies the Blaschke condition, that is, $\sum_j (1 - |a_j|) < \infty$. A finite or infinite Blaschke product is an inner function with zeros given by $\{ a_j \}$. We remark that the zero set of a holomorphic function in $\mathbb{D}$ satisfies the Blaschke condition.

Let μ be a positive finite measure on $\mathbb{T}$. Suppose in addition that μ is singular with respect to the Lebesgue measure, and set

$$s_\mu(z) = \exp\left(- \int_{\mathbb{T}} \frac{\psi + z}{\psi - z} \, d\mu(\psi) \right), \quad z \in \mathbb{D}.$$

It turns out that s_μ is an inner function and $s_\mu(z) \neq 0$ on $\mathbb{D}$. A function of this form is known as a singular inner function.

The *canonical factorization* of a function $f \in H^p$ states that there exists a unique factorization $f = \lambda b s_\mu f_{out}$, where $\lambda \in \mathbb{C}$, $|\lambda| = 1$, b is a Blaschke product associated with the zero set of f, s_μ is a singular inner function and f_{out} is the outer part of f.

Let C denote the algebra of continuous functions on $\mathbb{T}$. The *Sarason algebra* is the following algebraic sum

$$H^\infty + C = \{ f + g \, : \, f \in H^\infty, \, g \in C \}.$$

It is proved that this is indeed a closed sub-algebra of L^∞. The harmonic extension $\hat{\varphi}$ to $\mathbb{D}$ of a function $\varphi \in H^\infty + C$ is well-defined, and it plays a fundamental role in the characterization of invertible functions in this algebra. For $\varphi \in H^\infty + C$ and $0 < r < 1$, set $\varphi_r(e^{it}) = \hat{\varphi}(re^{it})$. Then φ is invertible in $H^\infty + C$ if and only if there exist $\delta, \epsilon > 0$ such that $|\varphi_r(e^{it})| \geq \epsilon$ for $1 - \delta < r < 1$ and $e^{it} \in \mathbb{T}$.

This criterion allows to define the index of an invertible function in $H^\infty + C$. For a non-vanishing function $\varphi \in C$, let $ind(\varphi) \in \mathbb{Z}$ be the index (or winding number) of φ around $z = 0$, which for differentiable φ can be computed as

$$ind(\varphi) = \frac{1}{2\pi i} \oint \frac{\varphi'}{\varphi} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\varphi'(e^{it})}{\varphi(e^{it})} e^{it} dt.$$

For φ is invertible in $H^\infty + C$, set $ind(\varphi) = \lim_{r\to 1^-} ind(\varphi_r)$. This index is stable by small perturbations and it is an homomorphism of the invertible functions in $H^\infty + C$ onto the group of integers. The key property to prove these facts as well as the criterion for invertibility is that the harmonic extension is asymptotically multiplicative in $H^\infty + C$.

The largest C^*-algebra of $H^\infty + C$ is the set of *quasicontinuous functions*

$$QC = (H^\infty + C) \cap (\overline{H^\infty + C})$$

Every unimodular $\theta \in QC$ is invertible in $H^\infty + C$. In [31] Sarason proved that each unimodular function $\theta \in QC$ of index $n \in \mathbb{Z}$ can be expressed as $\theta = \chi_n e^{i(u+\tilde{v})}$, where u, v are real functions in C and $\tilde{v}$ stands for the harmonic conjugate of v on $\mathbb{T}$.

Remark 2.1. In the case where φ is a rational function without zeros and poles on $\mathbb{T}$, it is well known that $ind(\varphi) = z - p$, being z and p the number of zeros and poles of φ in $\mathbb{D}$, respectively. More interesting, when φ is a unimodular function sufficiently regular (for instance if φ is of bounded variation), the index of φ can be computed using its Fourier coefficients $(\varphi_k)_{k\in\mathbb{Z}}$ as

$$ind(\varphi) = \sum_{k\in\mathbb{Z}} k\, |\varphi_k|^2;$$

see [10] and the references therein.

Fredholm index. The space of bounded linear operators on a Hilbert space H to a Hilbert space L is denoted by $\mathcal{B}(H, L)$ or $\mathcal{B}(H)$ if $H = L$. Let $\mathcal{K}(H, L) \subset \mathcal{B}(H, L)$ be the subspace of compact operators. Recall that an operator $A \in \mathcal{B}(H, L)$ is Fredholm if it has closed range and both its kernel and its cokernel $\operatorname{coker} A = L/\operatorname{Ran}(A)$ have finite dimension. In that case the *Fredholm index* of A is

$$\operatorname{ind}(A) = \dim\ker A - \dim\operatorname{coker} A.$$

Operators on Hardy spaces. Let $H^2_- = \chi_{-1}\overline{H^2}$ be the orthogonal complement of the Hardy space H^2, and consider the orthogonal projections P_+ and P_- onto H^2 and H^2_-, respectively. Three special classes of bounded operators will be used in the sequel. For $\varphi \in L^\infty$, the *multiplication operator* $M_\varphi \in \mathcal{B}(L^2)$, $M_\varphi f = \varphi f$, where $f \in L^2$; the *Toeplitz operator* $T_\varphi \in \mathcal{B}(H^2)$, $T_\varphi f = P_+(\varphi f)$, where $f \in H^2$; and the *Hankel operator* $H_\varphi \in \mathcal{B}(H^2, H^2_-)$, $H_\varphi f = P_-(\varphi f)$, where $f \in H^2$.

Recall that the (unilateral) shift operator is given by M_{χ_1}. It will be useful to state some well-known results on invariant subspaces of the shift operator.

Theorem 2.2. *Suppose that E is a closed subspace of L^2 and $M_{\chi_1}E \subseteq E$.*

i) (Wiener) If E is doubly invariant (i.e. $M_{\chi_1}(E) = E$), then $E = \chi_R L^2$ for a unique measurable subset $R \subseteq \mathbb{T}$, where χ_R is the characteristic of R.
ii) (Beurling–Helson) If E is singly invariant (i.e. $M_{\chi_1}(E) \neq E$), then $E = \theta H^2$ for a unique up to a constant $\theta \in L^\infty$ with $|\theta| = 1$ a.e.
iii) (Beurling) If $0 \neq E \subset H^2$, then $E = \theta H^2$ for some inner function θ.

We will frequently use several properties of Toeplitz operators. Among the basic properties we recall that $\|T_\varphi\| = \|\varphi\|_\infty$, $T_\varphi^* = T_{\overline{\varphi}}$ and $T_{\varphi\psi} = T_\varphi T_\psi$ whenever $\psi \in H^\infty$. The following results will be useful.

Theorem 2.3. *(Coburn's lemma) If $\varphi \in L^\infty$, then either* $\ker(T_\varphi) = \{0\}$ *or* $\ker(T_\varphi^*) = \{0\}$, *unless $\varphi \equiv 0$.*

Theorem 2.4. *Let φ be a function in L^∞. The following hold.*

i) T_φ is invertible if and only if it is Fredholm and has index zero.
ii) If $\varphi \in H^\infty + C$, then T_φ is Fredholm if and only if φ is invertible in $H^\infty + C$. Furthermore, the Fredholm index of T_φ satisfies $ind(T_\varphi) = -ind(\varphi)$.

3. The Grassmann manifold of L^2

Let Gr be the Grassmann manifold of L^2, i.e. the set of all closed subspaces of L^2. Let P_W denote the orthogonal projection onto a closed subspace $W \subset L^2$. In particular, we write $P_\varphi = P_{\varphi H^2}$, when $\varphi \in L^\infty$ and φH^2 is closed. If we identify each subspace with its orthogonal projection, then

$$Gr = \{\, P_W \,:\, W \text{ is a closed subspace of } L^2 \,\}.$$

Now we determine when φH^2 belongs to Gr.

Lemma 3.1. *Let φ be a nonzero function in L^∞. Then φH^2 is closed in L^2 if and only if φ is invertible in L^∞.*

Proof. Clearly, if the function φ is invertible in L^∞, then the subspace φH^2 is closed. Conversely, assume that the function φ is not invertible in L^∞. We first suppose that $\varphi \neq 0$ a.e. For $n \geq 1$, $k = 0, \ldots, 2^n - 1$, we define the following subsets of $\mathbb{T}$

$$E_{n,k} = \left\{ e^{2\pi i t} \,:\, t \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right),\ 1/\sqrt{2}^n \leq |\varphi(e^{2\pi i t})| \leq 1/\sqrt{2}^{n-1} \right\}.$$

Since φ is not invertible, there are infinitely many integers $n_1 < n_2 < \ldots$ and (k_j) such that $m(E_{n_j,k_j}) > 0$. Set $A = \cup_{j=1}^\infty E_{n_j,k_j}$, we define a function φ_0 as φ on A and as 1 on A^c. Note that $\varphi_0 \in L^\infty$ is not invertible and $\varphi_0 \neq 0$ a.e. We now take another function $f \in L^2$ such that $|f| \geq c > 0$ and $f/\varphi_0 \in L^1 \setminus L^2$. For instance, the function

$$f = \varphi_0 \left(\sum_{j=1}^{\infty} \frac{1}{\sqrt{m(E_{n_j,k_j})}} \chi_{E_{n_j,k_j}} + \chi_{A^c} \right)$$

satisfies these conditions. At this point, we remark that we had to introduce the function φ_0 to construct f: a function f satisfying the above properties with the given φ in place of φ_0 does not always exist (for instance if $1/\varphi \notin L^1$). Next define g_n as f/φ_0 on the set where $|\varphi_0| > 1/n$ and as 1 elsewhere. Denote by G_n the outer function satisfying $|G_n| = |g_n|$ (see e.g. [27, Thm. 3.9.1]). Note that the sequence $(G_n)_n$ converges in L^1 to a function $F \in H^1 \setminus H^2$ such that $|F| = |f/\varphi_0|$. The sequence $(\varphi_0 G_n)_n$ of functions in $\varphi_0 H^2$ converges in L^2 to $\varphi_0 F$, because it converges in L^1 and the sequence $(|\varphi_0 G_n|)_n$ is majorated by $|f| + 1 \in L^2$. It is left to notice $\varphi_0 F \notin \varphi_0 H^2$ because $F \notin H^2$. Hence we have shown that $\varphi_0 H^2$ is not closed. Note that the function s defined as 1 on A and φ on A^c is bounded and $s\varphi_0 = \varphi$. Then we can use the same sequence $(G_n)_n$ in H^2 to deduce that φH^2 is not closed.

Now suppose that $\varphi = 0$ in a subset $S \subseteq \mathbb{T}$ with positive measure. Assume that S is maximal with this property. We can reduce this case to the previous one. Take ψ a bounded function in S, which is not invertible and $\psi \neq 0$ a.e. Then define φ_1 as φ in S^c and as ψ on S. Since $\varphi_1 \neq 0$ a.e., we have a sequence $(G_n)_n$ in H^2 such that $\varphi_1 G_n$ converges to $\varphi_1 F$ in L^2, $F \notin H^2$. To finish the proof, note that $\varphi_1 \chi_S = \varphi$, and thus, the same sequence $(G_n)_n$ can be used to show that φH^2 is not closed. $\square$

We discuss briefly the geometry in the setting of an abstract C^*-algebra $\mathcal{A}$. Denote by $Gr(\mathcal{A})$ the Grassmann manifold of $\mathcal{A}$, i.e. the set of all self-adjoint projections in $\mathcal{A}$. In [29,13], Corach, Porta and Recht described the differential geometry of $Gr(\mathcal{A})$ in terms of projections and symmetries: one passes from projections to symmetries via the affine map

$$P \longleftrightarrow \epsilon_P = 2P - 1.$$

In [13] a natural reductive structure was introduced in $Gr(\mathcal{A})$. In particular, geodesics were characterized. In [29] it was proved that these geodesics have minimal length, if one measures the length of curves by

$$L(\alpha) = \int_0^1 \|\dot{\alpha}(t)\| \, dt,$$

where $\alpha : [0,1] \to Gr(\mathcal{A})$ is a piecewise C^1-curve and $\| \cdot \|$ is the norm of $\mathcal{A}$. This means that the operator norm induces a Finsler metric on $Gr(\mathcal{A})$; however, note that this metric is neither smooth, nor convex.

Let us summarize these facts in the following remark.

Remark 3.2. The Grassmann manifold $Gr(\mathcal{A})$ is a complemented submanifold of $\mathcal{A}$. Its tangent space $(TGr(\mathcal{A}))_P$ at P is given by

$$(TGr(\mathcal{A}))_P = \{ Y = iXP - iPX \,:\, X \in \mathcal{A}, X^* = X \},$$

which consists of self-adjoint operators which are co-diagonal with respect to P (i.e. $PYP = (I-P)Y(I-P) = 0$). Denote by $\mathcal{A}_h$ the space of self-adjoint elements of $\mathcal{A}$. A natural projection $E_P : \mathcal{A} \to (TGr(\mathcal{A}))_P$ is given by

$$E_P(X) = \text{co-diagonal part of } X = PX(I-P) + (I-P)XP.$$

This map induces a linear connection in $Gr(\mathcal{A})$: if $X(t)$ is a tangent field along a curve $\alpha(t) \in Gr(\mathcal{A})$,

$$\frac{DX}{dt} = E_\alpha(\dot{X}).$$

The geodesic of $Gr(\mathcal{A})$ starting at P with velocity Y has the form $\delta(t) = e^{t\tilde{Y}}Pe^{-t\tilde{Y}}$, where $\tilde{Y} = [Y, P]$ is antihermitian and co-diagonal with respect to P.

Let P, Q be two orthogonal projections such that $\|P - Q\| < 1$. Then there exists a unique operator $X \in \mathcal{A}_h$, with $\|X\| < \pi/2$, which is co-diagonal with respect to P, such that $Q = e^{iX}Pe^{-iX}$. The curve

$$\delta(t) = e^{itX}Pe^{-itX} \tag{1}$$

is the unique geodesic of $Gr(\mathcal{A})$ joining P and Q (up to reparametrization). Moreover, this geodesic has minimal length. The exponent X is an analytic function of P and Q:

$$X = -\frac{i}{2}\log(\epsilon_p \epsilon_Q),$$

which is an analytic logarithm because $\|\epsilon_P\epsilon_Q - 1\| = \|\epsilon_P - \epsilon_Q\| = 2\|P - Q\| < 2$.

More recently, necessary and sufficient conditions were given for the existence of a geodesic joining two given orthogonal projections in the Grassmann manifold $Gr(H)$ of a Hilbert space H. This includes the case in which $\|P - Q\| = 1$. To briefly describe this result, let us recall that Halmos [18] (see also [14,15]) proposed to understand the geometric properties of two orthogonal projections P and Q by considering the decomposition

$$(\text{Ran}(P) \cap \ker(Q)) \oplus (\text{Ran}(Q) \cap \ker(P)) \oplus (\text{Ran}(P) \cap \text{Ran}(Q)) \oplus (\ker(P) \cap \ker(Q)) \oplus H_0,$$

where H_0 is the orthogonal complement of the first four subspaces. The projections are said to be in *generic position* when the first four subspaces are trivial. The first two subspaces may be interpreted as an obstruction to find a geodesic between P and Q.

Remark 3.3. It was proved in [1] (see also [2]) that there is a geodesic (equivalently a minimal geodesic) in $Gr(H)$ joining P and Q if and only if

$$\dim \operatorname{Ran}(P) \cap \ker(Q) = \dim \operatorname{Ran}(Q) \cap \ker(P).$$

If both dimensions are equal to zero, then there exists a unique geodesic of minimal length in $Gr(H)$ joining P and Q. This geodesic has the same form as in (1) for a (unique) self-adjoint operator X satisfying $\|X\| \leq \pi/2$. In particular, note that there can be a unique minimizing geodesic even if $\|P - Q\| = 1$. If the above dimensions coincide but are nonzero, then there are infinitely many geodesics.

Returning to the study of subspaces of the form φH^2, we recall a well-known argument to reduce the injectivity problem of a Toeplitz operator with a general symbol to another one with unimodular symbol.

Suppose that φ is an invertible function in L^∞. Then there exists a function $\theta \in L^\infty$, $|\theta| = 1$ a.e., such that $\varphi H^2 = \theta H^2$. This gives a function $f \in H^2$ satisfying $\varphi = \theta f$. Note that f is invertible in L^∞. Since $\theta f H^2 = \varphi H^2 = \theta H^2$, it follows that $fH^2 = H^2$, and then, f is an outer function. Invertible functions in H^∞ are characterized as outer functions which are invertible in L^∞ (see e.g. [16, Prop. 7.34]). Then, f is an invertible function in H^∞, which clearly implies that the Toeplitz operator T_f is invertible. Since $f \in H^\infty$, it follows that $T_\varphi = T_\theta T_f$. Hence the kernel of T_φ is trivial if and only if the kernel of T_θ is trivial. Hence, if φ, ψ are invertible in L^∞, and $\varphi = \theta_1 f$, $\psi = \theta_2 g$ with θ_i inner and f, g outer, then $\varphi^{-1}\psi = \overline{\theta}_1 \theta_2 h$ where $h = f^{-1}g \in H^\infty$ is outer and invertible in L^∞ (invertible in H^∞). Therefore $T_{\varphi^{-1}\psi} = T_{\overline{\theta_1}\,\theta_2} T_h$ with T_h invertible, and $\ker(T_{\varphi^{-1}\psi}) = \ker(T_{\overline{\theta_1}\,\theta_2})$.

As a direct consequence of the above results, we can now relate the injectivity problem for Toeplitz operators with the problem of finding a geodesic between two given subspaces φH^2 and ψH^2.

Theorem 3.4. *Let φ, ψ be invertible functions in L^∞. The following are equivalent.*

i) $\ker(T_{\varphi\psi^{-1}}) = \ker(T_{\varphi^{-1}\psi}) = \{0\}$.
ii) There is a geodesic in Gr joining P_φ and P_ψ.
iii) There is unique geodesic of minimal length in Gr joining P_φ and P_ψ given by

$$\delta(t) = e^{itX} P_\varphi e^{-itX}, \quad t \in [0, 1],$$

where $X = X_{\varphi,\psi}$ is a uniquely determined self-adjoint operator such that $\|X\| \leq \pi/2$, $e^{iX} P_\varphi e^{-iX} = P_\psi$, which is co-diagonal with respect to both P_φ and P_ψ.

Proof. We can assume without loss of generality that φ, ψ are unimodular functions by the argument before the statement of this theorem. Then, note that the restriction of the multiplication operator

$$M_\psi|_{\ker(T_{\overline{\varphi}\psi})} : \ker(T_{\overline{\varphi}\psi}) \to (\varphi H^2)^\perp \cap \psi H^2,$$

is an isomorphism. Similarly, $\ker(T_{\varphi\overline{\psi}}) \simeq \varphi H^2 \cap (\psi H^2)^\perp$. If the kernels of both $T_{\varphi\overline{\psi}}$ and $T_{\overline{\varphi}\psi}$ are trivial, then by Remark 3.3 there is a geodesic joining P_φ and P_ψ. Conversely, if such a geodesic exists, then $\varphi H^2 \cap (\psi H^2)^\perp$ and $(\varphi H^2)^\perp \cap \psi H^2$ have the same dimension. By Coburn's lemma, this dimension must be zero. Thus, we have shown that the first and second item are equivalent. The equivalence between the second and third item is explained in Remark 3.3. $\square$

Remark 3.5. There are unimodular functions φ, ψ such that $\ker(T_{\varphi\overline{\psi}}) = \ker(T_{\overline{\varphi}\psi}) = \{0\}$ and $T_{\varphi\overline{\psi}}$ is not invertible. We exhibit a special class of such functions in Example 5.6.

3.1. On the operator $X_{\varphi,\psi}$

Let us study in more detail the self-adjoint operator $X = X_{\varphi,\psi}$ linking the subspaces φH^2 and ψH^2 in Theorem 3.4. To this effect, we recall the following facts concerning Halmos' model for two orthogonal projections P_0 and Q_0 in generic position acting in a Hilbert space H. Under this assumption, there exists an isometric isomorphism between H and a product space $K \times K$ and a positive operator Z in K with $\|Z\| \leq \pi/2$ and $\ker(Z) = \{0\}$. This isomorphism transforms the projections Q_0 and P_0 into

$$Q_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad P_0 = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix},$$

where $C = \cos(Z)$ and $S = \sin(Z)$ [18]. The unique self-adjoint operator X linking these projections is (see [1])

$$X = \begin{pmatrix} 0 & iZ \\ -iZ & 0 \end{pmatrix}.$$

Note that $\|X\| = \|Z\|$.

Let $\sigma(A)$ denote the spectrum of an operator A. Recall the definition of reduced minimum modulus $\gamma(A)$ of an operator $A \neq 0$:

$$\begin{aligned} \gamma(A) &= \inf\{\, \|Af\| \,:\, \|f\| = 1,\ f \in \ker(A)^\perp \,\} \\ &= \inf\, \sigma(|A|) \setminus \{0\}. \end{aligned}$$

Proposition 3.6. *Let φ, ψ be unimodular functions in L^∞ such that*

$$\ker(T_{\varphi\overline{\psi}}) = \ker(T_{\overline{\varphi}\psi}) = \{0\}.$$

Then

$$Z = M_\varphi \cos^{-1}\left(|T_{\varphi\overline{\psi}}|\right) M_{\overline{\varphi}}$$

and in particular

$$\|X_{\varphi,\psi}\| = \cos^{-1}(\gamma(T_{\varphi\overline{\psi}})).$$

Proof. On the non-generic part of P_φ and P_ψ, the operator $X = X_{\varphi,\psi}$ is trivial. Thus in order to compute its norm we restrict to the generic part, and thus X can be described by Halmos' model,

$$X = \begin{pmatrix} 0 & iZ \\ -iZ & 0 \end{pmatrix}.$$

It is elementary that, if Q_0, P_0 denote the reductions of P_φ, P_ψ to the generic parts, then

$$Q_0 P_0 Q_0 = \begin{pmatrix} C^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now

$$C^2 = P_\varphi P_\psi P_\varphi = M_\varphi P_+ M_{\overline{\varphi}} M_\psi P_+ M_{\overline{\psi}} M_\varphi P_+ M_{\overline{\varphi}} = M_\varphi T^*_{\varphi\overline{\psi}} T_{\varphi\overline{\psi}} M_{\overline{\varphi}} = M_\varphi |T_{\varphi\overline{\psi}}|^2 M_{\overline{\varphi}}.$$

Therefore $0 \le C = \cos(Z) = M_\varphi |T_{\varphi\overline{\psi}}| M_{\overline{\varphi}}$, and thus, $Z = M_\varphi \cos^{-1}\left(|T_{\varphi\overline{\psi}}|\right) M_{\overline{\varphi}}$. From this formula, it follows that

$$\|X_{\varphi,\psi}\| = \|\cos^{-1}(|T_{\varphi\overline{\psi}}|)\| = \cos^{-1}(\lambda_0),$$

where

$$\lambda_0 = \inf\, \sigma(|T_{\varphi\overline{\psi}}|) = \inf\, \sigma(|T_{\varphi\overline{\psi}}|) \setminus \{0\} = \gamma(T_{\varphi\overline{\psi}}).$$

The second equality can be deduced from the assumption that $T_{\varphi\overline{\psi}}$ is injective, which implies that 0 cannot be an isolated point of $\sigma(|T_{\varphi\overline{\psi}}|)$. $\square$

Example 3.7. Consider $\varphi = \chi_1$, where $\chi_1(z) = z$, and the Blaschke factor

$$\psi(e^{it}) = b_a(e^{it}) = \frac{\overline{a}}{|a|} \frac{a - e^{it}}{1 - \overline{a} e^{it}},$$

for $0 < |a| < 1$. Then by direct computation,

$$\varphi H^2 \cap (\psi H^2)^\perp = (\varphi H^2)^\perp \cap \psi H^2 = \{0\}\ ,\quad (\varphi H^2)^\perp \cap (\psi H^2)^\perp = H^2_-$$

and

$$\varphi H^2 \cap \psi H^2 = \chi_1 b_a H^2 = \chi_1(\chi_1 - a)H^2.$$

Then the generic part H_0 of φH^2 and ψH^2 is the two dimensional space $H^2 \ominus \chi_1(\chi_1 - a)H^2$. The reduced projections $Q_0 = P_\varphi|_{H_0}$ and $P_0 = P_\psi|_{H_0}$ are one dimensional,

$$\mathrm{Ran}(Q_0) = H_0 \cap \chi_1 H^2 = \left\langle \frac{\chi_1}{1-\overline{a}\chi_1} \right\rangle\ ,\quad \mathrm{Ran}(P_0) = H_0 \cap (\chi_1 - a)H^2 = \left\langle \frac{\chi_1 - a}{1-\overline{a}\chi_1} \right\rangle.$$

According to Halmos' formulas,

$$Q_0 P_0 Q_0 = \begin{pmatrix} C^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Denote by f and g the normalizations of $\frac{\chi_1}{1-\overline{a}\chi_1}$ and $\frac{\chi_1 - a}{1-\overline{a}\chi_1}$, respectively. As usual, let $f_1 \otimes f_2$ be the rank one operator defined by $f_1 \otimes f_2(h) = \ < h, f_2 > f_1$. Then

$$Q_0 P_0 Q_0 = (f \otimes f)(g \otimes g)(f \otimes f) = | < f, g > |^2 f \otimes f.$$

Therefore,

$$\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} = | < f, g > | f \otimes f.$$

In this case $C = cos(Z)$ is a positive real number, and thus $Z = \cos^{-1}(| < f, g > |)$. Simple computations show that $| < f, g > | = (1 - |a|^2)^{1/2}$, which gives

$$Z = \cos^{-1}((1 - |a|^2)^{1/2}) = \sin^{-1}(|a|).$$

Then, the part of $X_{\varphi,\psi}$ acting on H_0 is

$$X_{\varphi,\psi}|_{H_0} = \begin{pmatrix} 0 & -i \sin^{-1}(|a|) \\ i \sin^{-1}(|a|) & 0 \end{pmatrix}.$$

The restriction of $X_{\varphi,\psi}$ to $H_0^\perp$ is trivial. Thus, $X_{\varphi,\psi}$ has rank two, and

$$\|X_{\varphi,\psi}\| = \sin^{-1}(|a|).$$

3.2. Norm inequalities

The minimality property of the geodesics in the Grassmann manifold may be used to obtain operator inequalities.

Theorem 3.8. *Let φ, ψ be unimodular functions in L^∞ such that $\ker(T_{\varphi\overline{\psi}}) = \ker(T_{\overline{\varphi}\psi}) = \{0\}$. Then*

$$\|M_\theta P_+ - P_+ M_\theta\| \geq \cos^{-1}(\gamma(T_{\varphi\overline{\psi}})),$$

for every real argument $\theta \in L^\infty$ of the function $\varphi\overline{\psi}$.

Proof. Let θ be a real function in L^∞ such that $e^{i\theta} = \varphi\overline{\psi}$. Consider the curve

$$\alpha(t) = M_{e^{it\theta}} P_\varphi M_{e^{-it\theta}}.$$

Apparently, $\alpha(t)$ is a smooth curve in Gr with $\alpha(0) = P_\varphi$ and $\alpha(1) = M_{\overline{\varphi}\psi} P_\varphi M_{\varphi\overline{\psi}} = P_\psi$. Then $\alpha(t)$ is longer than the (unique) minimal geodesic which joins φH^2 and ψH^2, whose length is $\|X_{\varphi,\psi}\|$. Note that

$$\dot{\alpha}(t) = iM_{e^{it\theta}} M_\theta P_\varphi - iP_\varphi M_\theta M_{e^{-it\theta}} = iM_{e^{it\theta}} M_\varphi (M_\theta P_+ - P_+ M_\theta) M_{\overline{\varphi}} M_{e^{-it\theta}} .$$

Thus, we find that $\|\dot{\alpha}(t)\| = \|M_\theta P_+ - P_+ M_\theta\|$, and using Proposition 3.6, we obtain

$$\cos^{-1}(\gamma(T_{\varphi\overline{\psi}})) = \|X_{\varphi,\psi}\| \leq L(\alpha) = \int_0^1 \|\dot{\alpha}(t)\| dt = \|M_\theta P_+ - P_+ M_\theta\|. \quad \square$$

Remark 3.9. With the same hypothesis and notations as in the above theorem, note that the operator M_θ is self-adjoint. Therefore the commutator $[M_\theta, P_+] = M_\theta P_+ - P_+ M_\theta$ is antihermitian. Also elementary computations show that

$$P_+[M_\theta, P_+]P_+ = P_-[M_\theta, P_+]P_- = 0,$$

i.e. $[M_\theta, P_+]$ is co-diagonal with respect to P_+. Thus, its norm can be related to the norm of the Hankel operator H_θ by

$$\|[M_\theta, P_+]\| = \|P_- M_\theta P_+\| = \|H_\theta\|.$$

Then, by Nehari's theorem (see for instance [26]),

$$\|[M_\theta, P_+]\| = \inf\{\|\theta - f\|_\infty : f \in H^\infty\}.$$

Hence,

$$\|X_{\varphi,\psi}\| \leq \inf\{\|\theta - f\|_\infty : f \in H^\infty\}.$$

Special cases of the above inequality can be rephrased without any mention to complex unimodular functions.

Corollary 3.10. *Let θ be a real valued continuous function, then*

$$\|M_\theta P_+ - P_+ M_\theta\| \geq \cos^{-1}(\gamma(T_{e^{i\theta}})).$$

Proof. Put $\varphi = e^{i\theta}$ and $\psi = 1$ in Theorem 3.8. Then, note that φ is an invertible continuous function with zero index. Hence the operator T_φ is Fredholm and has index zero, which implies that it is invertible. $\square$

Let θ_t, $t \in [0,1]$, be a piecewise differentiable path of real valued functions in C. Then the curve $\alpha(t) = M_{e^{i\theta_t}} P_+ M_{e^{-i\theta_t}}$ is piecewise differentiable. Similarly as above, its velocity is

$$\|\dot{\alpha}(t)\| = \|M_{e^{i\theta_t}}[M_{i\dot{\theta}_t}, P_+]M_{-e^{i\theta_t}}\| = \|H_{\dot{\theta}_t}\| = \inf\{\|\dot{\theta}_t - f\|_\infty : f \in H^\infty\}.$$

The last quantity can be regarded as the norm of $[\dot{\theta}_t]$, the class of $\dot{\theta}_t$ in the quotient L^∞/H^∞ (which is also the velocity of the curve $[\theta_t]$ in the quotient). Therefore,

$$L(\alpha) = L_{L^\infty/H^\infty}([\theta_t]).$$

Note that the curve θ_t is arbitrary between θ_0 and θ_1. In particular, when θ_t is a straight line, we have the following:

Corollary 3.11. *Let θ_0, θ_1 be real valued continuous functions, then*

$$\|\theta_0 - \theta_1\|_{L^\infty/H^\infty} \geq \|X_{e^{i\theta_0}, e^{i\theta_1}}\| = \cos^{-1}(\gamma(T_{e^{i(\theta_1 - \theta_o)}})).$$

4. The action of $H^\infty + C$ on Gr_{res}

The space L^2 has the orthogonal decomposition $L^2 = H^2 \oplus H^2_-$, which we now use to give the following definition. The *compact restricted Grassmannian Gr_{res}* is the manifold of closed linear subspaces $W \subset L^2$ such that

- $P_+|_W : W \to H^2 \in \mathcal{B}(W, H^2)$ is a Fredholm operator, and
- $P_-|_W : W \to H^2_- \in \mathcal{B}(W, H^2_-)$ is a compact operator.

The components of the restricted Grassmannian are parametrized by $k \in \mathbb{Z}$, where k is the index of the operator $P_+|_W : W \to H^2 \in \mathcal{B}(W, H^2)$,

$$Gr^k_{res} = \{W \in Gr_{res} : ind(P_+|_W : W \to H^2) = k\}.$$

In particular, since P_+ is the identity restricted to H^2, $H^2 = \mathrm{Ran}(P_+) \in Gr^0_{res}$.

Lemma 4.1. *Let φ be an invertible function in L^∞. Then the following are equivalent.*

i) $\varphi H^2 \in Gr_{res}$.
ii) *φ is an invertible function in $H^\infty + C$.*
iii) *$\varphi H^2 = \theta H^2$ for some $\theta \in QC$, $|\theta| = 1$ a.e.*

In this case, $\varphi H^2 \in Gr^k_{res}$, where $k = -ind(\varphi) = -ind(\theta)$.

Proof. We first prove $i) \Rightarrow ii)$. We claim that the Hankel operator $H_\varphi : H^2 \to H^2_-$, $H_\varphi f = P_-(\varphi f)$, is compact if and only if $P_-|_{\varphi H^2} : \varphi H^2 \to H_-$ is compact. In fact, note that $H_\varphi f = P_-|_{\varphi H^2}(\varphi f) = P_-|_{\varphi H^2} M_\varphi f$, for all $f \in H^2$. Since φ is invertible in L^∞, $M_\varphi : H^2 \to \varphi H^2$ is an invertible operator. Thus,

$$H_\varphi = (P_-|_{\varphi H^2})(M_\varphi|_{H^2}), \quad H_\varphi (M_\varphi|_{H^2})^{-1} = P_-|_{\varphi H^2},$$

which clearly implies our claim.

Suppose that $\varphi H^2 \in Gr_{res}$. Then, the operator $P_-|_{\varphi H^2} : \varphi H^2 \to H^2_-$ is compact, so we get that H_φ is compact. Hartman's theorem asserts that a Hankel operator H_φ is compact if and only if $\varphi \in H^\infty + C$ (see e.g. [27, Thm. 2.2.5]). Thus, it follows that $\varphi \in H^\infty + C$. Since $\varphi H^2 \in Gr_{res}$, we also have that $P_+|_{\varphi H^2} : \varphi H^2 \to H^2$ is a Fredholm operator. Note that $\mathrm{Ran}(P_+|_{\varphi H^2}) = \mathrm{Ran}(T_\varphi)$ and $\ker(P_+|_{\varphi H^2}) = M_\varphi \ker(T_\varphi)$, where T_φ is the Toeplitz operator with symbol φ. Therefore T_φ is Fredholm, and thus, φ is invertible in $H^\infty + C$.

Now we prove $ii) \Rightarrow i)$. Assume that φ is an invertible function in $H^\infty + C$. Then, we have that T_φ is a Fredholm operator. By the same arguments as in the previous paragraph, we see that $P_+|_{\varphi H^2} : \varphi H^2 \to H^2$ is also a Fredholm operator. On the other hand, $\varphi \in H^\infty + C$ is equivalent to H_φ compact. Hence $P_-|_{\varphi H^2} : \varphi H^2 \to H_-$ is compact, and consequently, $\varphi H^2 \in Gr_{res}$.

The implication $ii) \Rightarrow iii)$ is given by Theorem 2.2: if $\varphi \in H^\infty + C$, then φH^2 is singly invariant. Therefore exists a (unique up to a multiplicative constant) unimodular function θ such that $\varphi H^2 = \theta H^2$. Now $\theta = \varphi f$ for some $f \in H^2$. Since φ is invertible in L^∞, then $f \in H^\infty$. Hence, $\theta \in H^\infty + C$. Further, by the invertibility of φ, it clearly follows that f is invertible in L^∞. Using that $\varphi H^2 = \theta H^2 = \varphi f H^2$, we get $f H^2 = H^2$, and consequently, f is an outer function. Recall that a function in H^∞ is invertible if and only if it is outer and invertible in L^∞. This gives $f^{-1} \in H^\infty$. Now $\overline{\theta} = \theta^{-1} = \varphi^{-1} f^{-1} \in H^\infty + C$, which proves that $\theta \in QC$.

To prove the implication $iii) \Rightarrow ii)$, we observe that every unimodular $\theta \in QC$ is invertible in $H^\infty + C$. By the equivalence between $i)$ and $ii)$, we get $\varphi H^2 = \theta H^2 \in Gr_{res}$, and hence φ is invertible in $H^\infty + C$.

Suppose that $\varphi H^2 \in Gr^k_{res}$. To prove our claim on the index, we have pointed out that $\mathrm{Ran}(P_+|_{\varphi H^2}) = \mathrm{Ran}(T_\varphi)$ and $\ker(P_+|_{\varphi H^2}) = M_\varphi \ker(T_\varphi)$, where M_φ is invertible. It follows that $k = ind(P_+|_{\varphi H^2}) = ind(T_\varphi) = -ind(\varphi)$. Moreover, $\theta = \varphi f$, and f is invertible in H^∞. Every invertible function in H^∞ has index zero. Hence, $ind(\varphi) = ind(\theta)$. $\square$

Under the identification of each closed subspace $W \subseteq L^2$ with the orthogonal projection P_W, the compact restricted Grassmannian is given by

$$Gr_{res} = \{\, P \in \mathcal{B}(L^2) \,:\, P - P_+ \text{ is compact}, \, P = P^2 = P^* \,\}. \tag{2}$$

Applying the results mentioned in Remark 3.2 for the algebra of compact operators, it follows that the tangent space $(TGr_{res})_P$ at some point $P \in Gr_{res}$ is given by

$$(TGr_{res})_P = \{\, iXP - iPX \,:\, X^* = X \text{ is compact} \,\}.$$

Then, using the usual operator norm, we have a Finsler metric to measure the length of curves.

On the other hand, the above presentation of Gr_{res} by means of operators is related to the orthogonal projections of the C^*-algebra

$$\mathcal{B}_{cc} = \{\, T \in \mathcal{B}(L^2) \,:\, [T, P_+] \text{ is compact} \,\}. \tag{3}$$

Indeed, this algebra consists of operators with compact co-diagonal entries. Recall that the Calkin algebra is the C^*-algebra obtained by taking the quotient of the algebra of bounded operators by the ideal of compact operators. Denoting by π the projection onto the Calkin algebra, the restricted Grassmannian coincides with the class of orthogonal projections $P \in \mathcal{B}_{cc}$ such that

$$\pi(P) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

where this is a matrix decomposition with respect to $\pi(P_+)$ and $\pi(P_-)$. Metric aspects of the projections in $\mathcal{B}_{cc}$ for a general Hilbert space H were studied in [5]. In particular, it was proved that any pair of projections in the same connected component of Gr_{res} can be joined by a geodesic of minimal length. Combining these facts and the characterization in Lemma 4.1, we have the following result.

Theorem 4.2. *Let φ, ψ be invertible functions in $H^\infty + C$. The following are equivalent.*

i) *$ind(\varphi) = ind(\psi)$.*
ii) *There is a geodesic in Gr_{res} joining P_φ and P_ψ.*
iii) *There is unique geodesic of minimal length in Gr_{res} joining P_φ and P_ψ given by*

$$\delta(t) = e^{itX} P_\varphi e^{-itX}, \quad t \in [0, 1],$$

where $X = X_{\varphi,\psi}$ is a uniquely determined compact self-adjoint operator such that $\|X\| < \pi/2$, $e^{iX} P_\varphi e^{-iX} = P_\psi$, and it is co-diagonal with respect to both P_φ and P_ψ.

Proof. We first show the equivalence between *i*) and *ii*). Suppose that $ind(\varphi) = ind(\psi)$, so we have that P_φ and P_ψ belong to the same connected component of Gr_{res}. According to [5, Thm. 6.6] there is a (minimal) geodesic joining these projections. The converse is obvious by the characterization of the connected components of Gr_{res} in terms of the index of the functions.

Similarly, to prove the equivalence between *i*) and *iii*), the only nontrivial part is that *i*) implies *iii*). If $ind(\varphi) = ind(\psi)$, then $ind(\varphi\psi^{-1}) = 0$, and consequently, as we state in Theorem 2.4, $T_{\varphi\psi^{-1}}$ is an invertible operator. Following the same argument as in the proof of Theorem 3.4, but now using Lemma 4.1, we can assume that φ, ψ are unimodular functions in QC. Therefore, $\varphi H^2 \cap (\psi H^2)^\perp \simeq \ker(T_{\varphi\overline{\psi}}) = \{0\}$ and $\psi H^2 \cap (\varphi H^2)^\perp \simeq \ker(T_{\psi\overline{\varphi}}) = \{0\}$. Under these conditions, there is a unique geodesic of minimal length joining P_φ and P_ψ of the desired form (see [5, Prop. 6.5, Thm. 6.6]). □

Remark 4.3. As we have seen in the proof, the above conditions are now equivalent to the invertibility of $T_{\varphi\psi^{-1}}$. The invertibility problem for Toeplitz operators has been well studied, see for instance the Widom–Devinatz theorem in [8, Thm. 2.23], and [20, Section 2, Thm. 5] for more related results.

4.1. Representation by continuous unimodular functions

Now we address the following question: when can we take the quasicontinuous function θ in Lemma 4.1 to be continuous? Note that this function is unique up to a multiplicative constant.

The conditions in Lemma 4.1 are also equivalent to have $\varphi H^2 = gH^2$, where $g \in C$ is non-vanishing. Indeed, this is easily seen from [26, Corollary 165.50.1], which asserts that the invertibility of a function φ in the algebra $H^\infty + C$ is equivalent to the factorization $\varphi = fg$, where $f, f^{-1} \in H^\infty$ and $g, g^{-1} \in C$. In addition, note that $ind(g) = ind(\varphi)$. However, the function g is not necessary unimodular.

Assuming that the function φ is continuous, we establish below a relation between θ and φ. Given a real valued function $u \in L^2$, $\tilde{u}$ is the harmonic conjugate on $\mathbb{T}$. Denote by Lip^α the Banach space of complex-valued functions on $\mathbb{T}$ satisfying a Lipschitz condition of order α $(0 < \alpha \leq 1)$. We write $A = H^\infty \cap C$ for the disk algebra.

Proposition 4.4. *Let $\varphi \in C$ be non-vanishing, θ denote the quasicontinuous function of Lemma 4.1, and set $u = -\log|\varphi|$, then*

$$\theta = \frac{\varphi}{|\varphi|} e^{i\tilde{u}}.$$

In particular, $\theta \in C$, whenever $\tilde{u} \in C$. In addition, the following assertions hold.

i) *If $\varphi \in Lip^\alpha$ for $0 < \alpha < 1$, then $\theta \in Lip^\alpha$.*
ii) *If $\varphi \in A$, then $\theta \in A$.*

Proof. Recalling that $\theta H^2 = \varphi H^2$, and by the proof of $ii) \Rightarrow iii)$ in Lemma 4.1, one can find an invertible function f in H^∞ such that $\theta = f\varphi$. Since f is an outer function, its harmonic extension admits a representation:

$$\hat{f}(z) = \lambda \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log|f(e^{it})| dt\right), \quad z \in \mathbb{D},$$

for some $\lambda \in \mathbb{T}$; see [26, Thm 3.9.6]. We may assume that $\lambda = 1$. Note that $\hat{f} = \exp(a+ib)$ where

$$a(z) = \frac{1}{2\pi}\int_0^{2\pi} Re\left\{\frac{e^{it}+z}{e^{it}-z}\right\} \log|f(e^{it})| dt = \log|\hat{f}(z)|,$$

since the real part of $(e^{it}+z)(e^{it}-z)^{-1}$ is the Poisson kernel. Since $|f| = 1/|\varphi|$ on $\mathbb{T}$, and $f \in H^\infty$, the following radial limit $\lim_{r\to 1^-} a(re^{it}) = \log|f(e^{it})| = u(e^{it})$ exists a.e. On the other hand,

$$b(z) = \frac{1}{2\pi}\int_0^{2\pi} Im\left\{\frac{e^{it}+z}{e^{it}-z}\right\} \log|f(e^{it})| dt$$

is the harmonic conjugate of a on $\mathbb{D}$ (up to a constant). By the Privalov–Plessner theorem [28, Thm. 6.1.1], $\lim_{r\to 1^-} b(re^{it}) = \tilde{u}(e^{it})$ a.e. Since $\theta = \varphi f$ and $f = e^u e^{i\tilde{u}} = \frac{1}{|\varphi|} e^{i\tilde{u}}$, we obtain $\theta = \frac{\varphi}{|\varphi|} e^{i\tilde{u}}$.

$i)$ Now we assume that $\varphi \in Lip^\alpha$. Since φ is a non-vanishing continuous function, then $u = -\log|\varphi| \in Lip^\alpha$. By Privalov's theorem, $\tilde{u} \in Lip^\alpha$ for $\alpha < 1$ (see [28, Thm. 10.1.3]). Clearly, $\varphi, |\varphi|^{-1} \in Lip^\alpha$, which yields $\theta \in Lip^\alpha$.

$ii)$ According to [27, Section 4.3.8], the outer part φ_{out} of φ belongs to A. Since $\theta = \varphi f$, it follows that $|f^{-1}| = |\varphi_{out}|$. Therefore, $\varphi_{out} = \lambda f^{-1}$ for some $\lambda \in \mathbb{T}$. Thus, the inner part of φ satisfies $\theta = \lambda \varphi_{inn}$, and thus we obtain $\theta \in A$. $\square$

Example 4.5. In contrast to what happens with functions in Lip^α or A, we now show that the class of absolutely continuous functions is not preserved in the above proposition. Let

$$u(e^{it}) = -\sum_{n\geq 2} \frac{\sin(nt)}{n\log(n)}$$

then $u \in C$; moreover u is absolutely continuous on $\mathbb{T}$ [36, p. 241]. Let $\varphi = e^{-u}$, clearly $\varphi \in C$ is non-vanishing and absolutely continuous on $\mathbb{T}$. Since $u(\mathbb{T}) \subset \mathbb{R}$, we have $\varphi > 0$ on $\mathbb{T}$, therefore $-\log|\varphi| = u$. Let

$$v(e^{it}) = \sum_{n\geq 2} \frac{\cos(nt)}{n\log(n)},$$

and note that

$$f(z) = \sum_{n\geq 2} \frac{i}{n\log(n)} z^n = iv + u$$

is analytic, therefore v is the harmonic conjugate of u. But v is not continuous on $\mathbb{T}$, not even bounded since $\sum_{n\geq 2} \frac{1}{n\log(n)} = +\infty$, therefore $\theta = e^{iv}$ is not continuous on $\mathbb{T}$.

4.2. p-Norms

Given an operator $T \in \mathcal{K}(H, L)$, we denote by $(s_n(T))_{n\geq 1}$ the sequence of its singular values. The p-Schatten class ($1 \leq p < \infty$) is defined by

$$\mathcal{B}_p(H, L) = \left\{ T \in \mathcal{K}(H, L) \,:\, \|T\|_p = \left(\sum_{n=1}^{\infty} s_n(T)^p \right)^{1/p} < \infty \right\}.$$

These are Banach spaces endowed with the norm $\|\cdot\|_p$. As usual, when $p = \infty$, we set $\mathcal{B}_\infty(H, L) = \mathcal{K}(H, L)$. In particular, $\mathcal{B}_p(H, H) = \mathcal{B}_p(H)$ is a bilateral ideal of $\mathcal{B}(H)$. Using the orthogonal decomposition $L^2 = H^2 \oplus H^2_-$, and the p-Schatten class ($1 \leq p < \infty$), one can introduce the *p-restricted Grassmannian* $Gr_{res,p}$ as the manifold of closed linear subspaces $W \subset L^2$ such that

- $P_+|_W : W \to H^2 \in \mathcal{B}(W, H^2)$ is a Fredholm operator, and
- $P_-|_W : W \to H^2_- \in \mathcal{B}_p(W, H^2_-)$.

Its connected components $Gr^k_{res,p}$, $k \in \mathbb{Z}$, are also described by the index of the projection $P_+|_W : W \to H^2$. The case $p = 2$ was studied in connection with loop groups [30]; it is an infinite dimensional manifold with remarkable geometric properties [7,17,35]. Other values of $1 \leq p \leq \infty$, or more generally restricted Grassmannians associated with symmetrically-normed ideals, were treated in [6,11].

We now introduce Besov spaces. For more details on the following definitions and results we refer to Böttcher, Karlovich and Silbermann [9]. The moduli of continuity of a function $f \in L^p$ are defined as follows: for $s > 0$,

$$\omega^1(f, s) = \sup_{|h|\leq s} \|f(e^{i(\,\cdot\,+h)}) - f(e^{i\,\cdot\,})\|_{L^p}\,;$$

$$\omega^2(f, s) = \sup_{|h|\leq s} \|f(e^{i(\,\cdot\,+h)}) - 2f(e^{i\,\cdot\,}) + f(e^{i(\,\cdot\,-h)})\|_{L^p}\,.$$

For $1 \le p < \infty$ and $0 < \alpha \le 1$, put

$$|f|_{B_p^\alpha} := \begin{cases} \left(\int_0^{2\pi} (s^{-\alpha}\omega^1(f,s))^p \, \frac{ds}{s} \right)^{1/p} & \text{if } \ 0 < \alpha < 1; \\ \left(\int_0^{2\pi} (s^{-\alpha}\omega^2(f,s))^p \, \frac{ds}{s} \right)^{1/p} & \text{if } \ \alpha = 1. \end{cases}$$

The *Besov space* B_p^α is defined as

$$B_p^\alpha := \{ f \in L^p \, : \, |f|_{B_p^\alpha} < \infty \} .$$

It is a Banach space endowed with the norm $\|f\|_{B_p^\alpha} := \|f\|_{L^p} + |f|_{B_p^\alpha}$. We denote by B_p^α the *Besov space*, where $1 \le p < \infty$ and $0 < \alpha \le 1$.

For another description of Besov spaces in terms of special kernels, see for instance [26, Appendix 5].

Among various generalizations of the classical Krein algebra, in [9] it was introduced the following algebra defined by means of Hankel operators:

$$K_{p,0}^{1/p,0} = \{ \varphi \in L^\infty \, : \, H_\varphi \in \mathcal{B}_p(H^2, H^2_-) \},$$

where $1 \le p \le \infty$. It turns out to be a Banach algebra under the norm

$$\|\varphi\|_{K_{p,0}^{1/p,0}} = \|\varphi\|_{L^\infty} + \|H_\varphi\|_p \, .$$

In the case $p = \infty$, it simply has the usual operator norm of a compact operator. By Hartman's theorem, $K_{\infty,0}^{1/\infty,0} = H^\infty + C$, and for $1 \le p < \infty$, one has $K_{p,0}^{1/p,0} \subseteq H^\infty + C$. Given a function $\varphi \in L^\infty$ and $1 \le p < \infty$, Peller's theorem states that the Hankel operator $H_\varphi \in \mathcal{B}_p(H^2, H^2_-)$ if and only if $P_-\varphi \in B_p^{1/p}$ (see [26, Thm. 1.1, Appendix 5]). Then there is an equivalent definition of $K_{p,0}^{1/p,0}$ in terms of functions instead of operators. When $1 \le p < \infty$, it holds

$$K_{p,0}^{1/p,0} = \{ \varphi \in L^\infty \, : \, P_-\varphi \in B_p^{1/p} \} = L^\infty \cap (H^\infty + B_p^{1/p}).$$

Moreover, when $p > 1$, a function φ is invertible in $K_{p,0}^{1/p,0}$ if and only if is invertible in $H^\infty + C$.

Using the above stated results and the same arguments of Lemma 4.1, the following characterization can be obtained.

Corollary 4.6. *Let φ be an invertible function in L^∞ and $1 \le p < \infty$. The following assertions are equivalent:*

i) $\varphi H^2 \in Gr_{res,p}$.
ii) $\varphi \in K_{p,0}^{1/p,0}$ *and* φ *is invertible in* $H^\infty + C$.
iii) $\varphi H^2 = \theta H^2$ *for some* $\theta \in QC \cap K_{p,0}^{1/p,0}$, $|\theta| = 1$ *a.e.*

In this case, $\varphi H^2 \in Gr_{res}^k$*, where* $k = -ind(\varphi) = -ind(\theta)$.

Remark 4.7. For $p > 1$, condition *ii)* can be replaced by

ii′) φ is an invertible function in $K_{p,0}^{1/p,0}$.

The description for the compact restricted Grassmannian given in (2) has an analogue for the p-restricted Grassmannian

$$Gr_{res,p} = \{\, P \in \mathcal{B}(L^2) \,:\, P - P_+ \in \mathcal{B}_p(L^2),\ P = P^2 = P^* \,\}.$$

The tangent space at $P \in Gr_{res,p}$ can be identified with

$$(TGr_{res,p})_P = \{\, iXP - iPX \,:\, X^* = X \in B_p(L^2) \,\} \subseteq \mathcal{B}_p(L^2).$$

Then, a natural Finsler metric is defined by using the p-norm, which gives the following length functional: for $\alpha : [0,1] \to Gr_{res,p}$ is a piecewise C^1-curve,

$$L_p(\alpha) = \int_0^1 \|\dot{\alpha}(t)\|_p \, dt.$$

The geodesics defined in (1) are also minimal for the p-norm (see [6, Cor. 27]). Thus, we can use the same ideas of Theorem 4.2 to prove the following (note that $ind(\varphi) = ind(\psi)$ forces $\|P_\varphi - P_\psi\| < 1$ by previous remarks):

Corollary 4.8. *Let* $1 \le p < \infty$*, and let* φ, ψ *be functions in* $K_{p,0}^{1/p,0}$ *which are invertible in* $H^\infty + C$*. The following are equivalent:*

i) $ind(\varphi) = ind(\psi)$.
ii) There is a geodesic in $Gr_{res,p}$ *joining* P_φ *and* P_ψ.
iii) There is unique geodesic of minimal length in $Gr_{res,p}$ *joining* P_φ *and* P_ψ *given by*

$$\delta(t) = e^{itX} P_\varphi e^{-itX}, \quad t \in [0,1],$$

where $X = X_{\varphi,\psi}$ *is a uniquely determined self-adjoint operator such that* $\|X\| < \pi/2$, $e^{iX} P_\varphi e^{-iX} = P_\psi$*, and it is co-diagonal with respect to both* P_φ *and* P_ψ.

Moreover, arguing as in the proof of Theorem 3.8 we also obtain

Corollary 4.9. *Let $1 \leq p < \infty$, and let φ, ψ be functions in $K_{p,0}^{1/p,0}$ which are invertible in $H^\infty + C$, such that $ind(\varphi) = ind(\psi)$. Then if $\theta \in K_{p,0}^{1/p,0}$ is such that $e^{i\theta} = \varphi\overline{\psi}$,*

$$\|M_\theta P_+ - P_+ M_\theta\|_p \geq 2^{1/p} \| \cos^{-1}(|T_{\varphi\overline{\psi}}|)\|_p = dist_p(P_\varphi, P_\psi).$$

For instance, if φ and ψ are C^1 functions (with equal index) such an argument θ exists, which is continuous and piecewise smooth.

Proof. Recall from Proposition 3.6 that

$$X_{\varphi,\psi} = \begin{pmatrix} 0 & iZ \\ -iZ & 0 \end{pmatrix}$$

and thus ($Z \geq 0$)

$$|X_{\varphi,\psi}| = \begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix}.$$

Also $Z = M_\varphi \cos^{-1}(|T_{\varphi\overline{\psi}}|) M_{\overline{\varphi}}$. Then

$$\|X_{\varphi,\psi}\|_p = 2^{1/p}\|Z\|_p = 2^{1/p}\| \cos^{-1}(|T_{\varphi\overline{\psi}}|)\|_p \quad \square$$

5. Shift-invariant subspaces of H^2

The orthogonal projections of the C^*-algebra $\mathcal{B}_{cc}$ defined in (3) may be classified using their image in the Calkin algebra. In addition to the restricted Grassmannian, we shall need to consider the essential class $\mathbb{E}_1$ consisting of all the orthogonal projections which have the form (in terms of $\pi(P_+)$ and $\pi(P_-)$)

$$\pi(P) = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix},$$

where $p \neq 0, 1$ is a projection in the Calkin algebra. It was shown that the class $\mathbb{E}_1$ is connected, and in contrast to the restricted Grassmannian, there are projections which cannot be joined by a geodesic in $\mathbb{E}_1$.

Let E be a closed subspace of L^2 such that $M_{\chi_1}(E) \subset E$. If $0 \neq E \subseteq H^2$, then $E = \varphi H^2$ for some inner function φ. We prove below that these subspaces belong to either the restricted Grassmannian or the essential class $\mathbb{E}_1$.

Theorem 5.1. *Let φ be an inner function. Then the following assertions hold:*

i) φ is a finite Blaschke product if and only if $P_\varphi \in Gr_{res}^k$, where k is the number of zeros of φ.
ii) φ is not a finite Blaschke product if and only if $P_\varphi \in \mathbb{E}_1$.

Proof. *i*) The only inner functions which are invertible in $H^\infty + C$ are the finite Blaschke products (see e.g. [34, Thm. 5]). Therefore, the result follows from Lemma 4.1. The index of a Blaschke factor is equal to its number of zeros (see Remark 2.1), and as we have already showed, it determines the connected component of Gr_{res} where P_φ lies.

ii) First recall that an orthogonal projection

$$P = \begin{pmatrix} x & a \\ a^* & y \end{pmatrix},$$

belongs to G_{res} if and only if a, y are compact operators and x is Fredholm (see [5, Lemma 3.3]). Now suppose that φ is not a finite Blaschke product. As we remarked in the preceding item, this means that φ is not invertible in $H^\infty + C$. Therefore, $P_\varphi \notin Gr_{res}$ by Lemma 4.1. On the other hand, note that $a^* = P_- P_\varphi|_{H^2} = 0$ and $y = P_- P_\varphi|_{H^2_-} = 0$. Using that $P_\varphi \notin Gr_{res}$, we obtain that $x = P_+ P_\varphi|_{H^2}$ is not Fredholm. In order to prove that $P_\varphi \in \mathbb{E}_1$, it only remains to verify that x is not compact. To this end, it suffices to show that $\dim \ker(x-1) = \infty$. But since $\varphi \in H^\infty$, we have $\ker(x-1) = H^2 \cap \varphi H^2 = \varphi H^2$, which has infinite dimension. The converse is an immediate consequence of Lemma 4.1 and the characterization of invertible inner functions in $H^\infty + C$. $\square$

Remark 5.2. Every geodesic in $\mathbb{E}_1$ is a geodesic in Gr. This follows by the explicit form of geodesics in a general C^*-algebra described in Remark 3.2. However, the converse does not hold: geodesics in Gr joining two projections of $\mathbb{E}_1$ may lie outside of $\mathbb{E}_1$. Suppose that $P, Q \in \mathbb{E}_1$, and there is a geodesic $\delta(t) = e^{itX} P e^{-itX}$ in Gr joining these projections. Then $\delta(t)$ belongs to $\mathbb{E}_1$ if and only if $X \in \mathcal{B}_{cc}$ (see [5, Prop 6.11] for other equivalent conditions).

5.1. Examples

We shall give examples of shift-invariant subspaces which can or cannot be joined by a (minimal) geodesic in the Grassmann manifold Gr. The simplest case is a consequence of the following result proved in [23, Lemma 3.2] for Hardy spaces of the upper half-plane. It is an elementary but important step to understand Toeplitz kernels. We shall state it for the Hardy space of the circle.

Lemma 5.3. *Let φ, ψ be two inner functions. Then $\ker(T_{\varphi\overline{\psi}}) \neq \{0\}$ if and only if there exist an inner function θ and an outer function g such that $\varphi\,\theta\,g = \psi\overline{g}$ on $\mathbb{T}$.*

Example 5.4. Suppose that φ divides ψ. This means that there is an inner function θ such that $\varphi\theta = \psi$. Thus, the equation in Lemma 5.3 is satisfied with $g = 1$, and consequently, $\ker(T_{\varphi\overline{\psi}}) \neq \{\,0\,\}$. Hence there is no geodesic in Gr joining φH^2 and ψH^2. Note that $\ker(T_{\overline{\varphi}\psi}) = \{\,0\,\}$. In this case, it is not difficult to construct concrete examples using the following well-known description of divisors in H^∞. Suppose that $\{a_j\}$ and $\{a'_j\}$ are the zero sets of φ and ψ, respectively. If $\varphi = \lambda b s_\mu$ and $\psi = \lambda' b' s_{\mu'}$ are the canonical factorizations, then φ divides ψ if and only if $\{a_j\} \subseteq \{a'_j\}$ and $\mu \leq \mu'$.

The canonical factorization also turns out to be relevant to give an affirmative answer to the existence of a geodesic in many concrete cases. Let φ be an inner function. A point on $\mathbb{T}$ belongs to the *support* of φ if it is a limit point of zeros of φ or if it belongs to the support of the singular measure associated with the singular factor of φ. We write *supp*(φ) for the support of φ. Sarason and Lee proved the following [22, Thm. 1–2].

Theorem 5.5. *Let φ, ψ be inner functions.*

i) If supp$(\varphi) \neq$ supp(ψ), then the spectrum of $T_{\varphi\overline{\psi}}$ is the closed unit disk.
ii) If there is a point $z_0 \in$ supp$(\psi) \setminus$ supp(φ), then $T_{\varphi\overline{\psi}} - \lambda$ has dense range for all λ.

From the above result and Theorem 3.4 we obtain this example.

Example 5.6. Let φ, ψ be inner functions. Suppose that there are two points z_0 and z_1 such that $z_0 \in$ *supp*$(\psi) \setminus$ *supp*(φ) and $z_1 \in$ *supp*$(\varphi) \setminus$ *supp*(ψ). Then there is unique minimal geodesic in Gr joining P_φ and P_ψ of the form stated in Theorem 3.4.

Now we consider the case of two inner functions with support $z = 1$. As a direct consequence of the results on Toeplitz kernels obtained by Makarov, Mitkovski and Poltoratski [23,25] (see also the survey [19]), one can show examples of the two inner functions of the aforementioned type such that their corresponding subspaces can or cannot be joined by a geodesic in Gr. These remarkable results were proved for Toeplitz operators in Hardy spaces of the upper-half plane (and other classes of functions). For this reason, we shall change to the half-plane; however by the isometry exhibited below all can be translated to the disk.

A function F holomorphic on the upper half-plane $\mathbb{C}_+ = \{\, z \,:\, Im\, z > 0 \,\}$ belongs to the Hardy space $H^2_+ = H^2(\mathbb{C}_+)$ if

$$\|F\|_{H^2_+} := \left(\sup_{y>0} \int_{-\infty}^{\infty} |F(x+iy)|^2 dx \right)^{1/2} < \infty.$$

As in Hardy spaces of the disk, one may consider H^2_+ as a Hilbert subspace of $L^2(\mathbb{R})$ since non-tangential limits exist a.e. No confusion will arise if we also denote by P_+ the orthogonal projection of $L^2(\mathbb{R})$ onto H^2_+. The Toeplitz operator with symbol $U \in L^\infty(\mathbb{R})$ is defined by

$$T_U : H^2_+ \to H^2_+, \quad T_U(F) := P_+(UF).$$

We write $H^\infty_+ = H^\infty(\mathbb{C}_+)$ for the bounded holomorphic functions on $\mathbb{C}_+$. Notice that $w = \frac{z-i}{z+i}$ is a conformal map from $\mathbb{C}_+$ onto $\mathbb{D}$. Set $f(w) = F(z)$. Then, it follows that $F(z) \in H^\infty_+$ if and only if $f(w) \in H^\infty$. However, H^2_+ is not obtained from H^2 by conformal mapping. It can be shown that $f(w) \in H^2$ if and only if $\frac{\pi^{-1/2}}{(z+i)} F(z) \in H^2_+$. Taking boundary values, one sees that

$$W : H^2 \to H^2_+, \quad Wf(x) = \frac{\pi^{-1/2}}{(x+i)} f\left(\frac{x-i}{x+i}\right), \;\; x \in \mathbb{R},$$

is an isometry from H^2 onto H^2_+. Set $\gamma(x) = \frac{x-i}{x+i}$ and fix $\theta \in L^\infty$. Then, Toeplitz operators in the Hardy spaces of the disk and the upper half-plane are related by

$$WT_\theta = T_{\theta\circ\gamma} W .$$

The canonical factorization of functions in H^2 can be also derived in H^2_+ using the isometry W.

By an inner function Θ in $\mathbb{C}_+$ we mean that $\Theta \in H^\infty_+$ and $|\Theta| = 1$ on $\mathbb{R}$. An inner function $\Theta(z)$ in $\mathbb{C}_+$ is a *meromorphic inner function* if it has a meromorphic extension to $\mathbb{C}$. In this case, the meromorphic extension to the lower half-plane is given by $\Theta(z) = \frac{1}{\Theta(\overline{z})}$. Each meromorphic inner function Θ admits a canonical factorization $\Theta = B_\Lambda S^a$, where $a \geq 0$ and Λ is a discrete set in $\mathbb{C}_+$ without accumulation points on $\mathbb{R}$ such that the following Blaschke condition holds

$$\sum_{\lambda\in\Lambda} \frac{Im\,\lambda}{1+|\lambda|^2} < \infty.$$

The function B_Λ is the corresponding Blaschke product in $\mathbb{C}_+$, i.e.

$$B_\Lambda(z) = \prod_{\lambda\in\Lambda} \epsilon_\lambda \frac{z-\lambda}{z-\overline{\lambda}}; \;\; |\epsilon_\lambda| = 1.$$

The other function in the factorization is given by the singular inner function $S^a(z) = e^{iaz}$. Meromorphic inner functions correspond to inner functions in H^2 such that $z = 1$ is the only possible accumulation point of their zeros and also the only possible singular point mass.

Example 5.7. The *point spectrum* of a meromorphic inner function $\Theta = B_\Lambda S^a$ is the set $\sigma(\Theta) = \{\, \Theta = 1 \,\}$ or $\{\, \Theta = 1 \,\} \cup \{\infty\}$. The point ∞ belongs to the spectrum if $\sum_{\lambda\in\Lambda} Im\,\lambda < \infty$ and $S^a \equiv 1$ (see [23] for other equivalent conditions). Two meromorphic inner functions are said to be *twins* if they have the same point spectrum, possibly including infinity. The twin inner function theorem asserts that if Θ, J are twins, then $\ker(T_{\overline{\Theta}J}) = \{0\}$ [23, Thm. 3.20]. Thus, there is always a geodesic joining the corresponding subspaces defined by twin functions.

Example 5.8. Recall that a sequence of real numbers is separated if $|\lambda_n - \lambda_m| \geq \delta > 0$ $(n \neq m)$. A separated sequence $(\lambda_n)_{n\in\mathbb{Z}}$ is a called a Pólya sequence if every zero-type entire function bounded on $(\lambda_n)_{n\in\mathbb{Z}}$ is constant (see also [25] for a new characterization). Among several conditions, it was proved in [25, Thm. A] that $(\lambda_n)_{n\in\mathbb{Z}}$ is a Pólya sequence if and only if there exists a meromorphic inner function Θ with $\{\Theta = 1\} = (\lambda_n)_{n\in\mathbb{Z}}$ such that $\ker(T_{\overline{\Theta}S^{2c}}) \neq \{0\}$ for some $c > 0$. Hence there is no geodesic joining the corresponding subspaces defined by Θ and S^{2c}.

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